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ON THE STRUCTURE OF A SOFIC SHIFT SPACE

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ABSTRACT. The structure of a sofic shift space is investigated, and Krieger's embedding theorem and Boyle's factor theorem are generalized to a large class of sofic shifts.

1. Introduction

In symbolic dynamics there are two celebrated results which give necessary and sufficient conditions for an irreducible subshift of finite type (SFT) to embed into or factor onto an irreducible SFT with different entropy. They are known as Krieger's embedding theorem and Boyle's factor theorem, and they are closely related: Their proofs are similar, being based on a fundamental lemma of Krieger, and they are both quite striking because they ensure that a certain trivial necessary condition is also sufficient. Furthermore, the necessary and sufficient conditions have to do with the periods of the periodic points in both cases; in Krieger's theorem the requirement is that the target shift must contain at least as many periodic orbits of each period as the domain shift, and in Boyle's theorem it is that the target must contain an orbit whose period divides n, whenever the domain contains an orbit of period n. When the target shift space is mixing, rather than merely irreducible, Krieger's theorem holds unchanged when the domain shift is no longer an SFT — in fact, the domain can be an arbitrary shift space as long as the range is a mixing SFT. Similarly, when the target is a mixing SFT the validity of Boyle's theorem extends far beyond the case where the domain is an SFT. In particular, it holds unchanged when the domain is an arbitrary sofic shift, as long as the target is a mixing SFT; cf. [B]. However, Boyle demonstrated by examples that both theorems fail quite dramatically when the target is allowed to be a mixing sofic shift, even when the domain is a mixing SFT. It is the purpose of the present paper to reveal and investigate the structure of a sofic shift space which is responsible for this failure, and to obtain necessary and sufficient conditions for the existence of embeddings and factor maps between irreducible sofic shifts of different entropies. As we shall explain below, we do not succeed completely in the latter, but we do make substantial progress by proving results which cover and go beyond the cases where the target is an arbitrary irreducible sofic shift and the domain is an irreducible SFT. Furthermore, we do have a good idea about what is lacking in order to cover the general case.

The fundamental difference between SFTs and other shift spaces has to do with synchronizing words. While a general shift space may not have any synchronizing

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words, in an SFT all sufficiently long words are synchronizing. This is no longer the case when the shift space is strictly sofic, but still there are many synchronizing words and this fact is of central importance in the study of sofic shift spaces and their presentations. The presence but relative scarcity of synchronizing words gives rise to a natural structure of layers in a sofic shift space which is absent in an SFT. Specifically, when X is a sofic shift space with non-wandering part $R(X)$, we can consider the shift space

$$\partial X = \{x \in R(X) : \ x \text{ contains no words that are synchronizing for } R(X)\},$$

which we call *the derived shift space* of X. It turns out that the derived shift space is again a sofic shift space, and we can therefore repeat the construction by considering the derived shift space $\partial^2 X$ of ∂X. By continuing in this way we obtain a decreasing sequence of subshifts inside a sofic shift:

$$X \supseteq \partial X \supseteq \partial^2 X \supseteq \partial^3 X \supseteq \dots .$$

It turns out that the sequence terminates, i.e. there is an $n \in \mathbb{N}$ such that $\partial^n X \neq \emptyset$ while $\partial^{n+1} X = \emptyset$. We call n the *depth* of X. The derived shift spaces of X, and hence also the depth of X, are conjugacy invariants of X, and any natural number n can occur as the depth of a mixing sofic shift space.

Even when X is mixing the derived shift spaces may not be irreducible, and we are therefore forced to deal with reducible sofic shift spaces in our study of mixing sofic shifts. This conclusion is not new; it was pointed out by Boyle and Krieger in [BK]. In order to understand the nature of reducibility occurring in irreducible sofic shift spaces, we generalize here the notion of irreducible components, which give rise to the fundamental structure in a reducible SFT, to make it more relevant for sofic shifts. As with the derived shift spaces, the point of departure for this is the set of synchronizing words in the non-wandering part $R(X)$ of X. Following Jonoska, [J2], we introduce an equivalence relation between these words so that two synchronizing words, x and y, which occur in $R(X)$, are equivalent exactly when there are words t, s in $R(X)$ so that xsy and ytx are words in $R(X)$. The set of equivalence classes parametrizes the collection of what we call the irreducible components, and for each equivalence class the corresponding irreducible component is made up by the elements in $R(X)$ in which there is a bound on the distance between occurrences of words from the equivalence class. Each irreducible component is a subshift of X, which may not be closed, but is the union of an increasing sequence of irreducible SFTs. They are mutually disjoint, and they are disjoint from the derived shift space ∂X of X. When X happens to be an SFT, the irreducible components are closed in X, because all sufficiently long words are synchronizing, and they agree with the usual irreducible components in this case. Unlike the SFT case, the irreducible components, as defined above, do not contain all periodic points of a general sofic shift space. But this is just because ∂X contains periodic points, so to capture more periodic points into irreducible components we add first the irreducible components of ∂X, then the irreducible components of $\partial^2 X$, and so on. It turns out that each of the derived shift spaces of X only contributes a finite number of new irreducible components, and as a result the total number of irreducible components we obtain is finite because the depth is finite. The new collection of irreducible components, coming from all the layers in X, does contain all periodic points, and they are mutually disjoint, as they are in the SFT case. The fundamental difference is that they may not be closed in X, but their closure is always an irreducible sofic subshift of X. This structure of irreducible components is a conjugacy invariant for the shift space, and it is a structure which is respected by any map between sofic shift spaces. Specifically, when X and Y are sofic shift spaces with irreducible components $X_c, c \in C_X$, and $Y_c, c \in C_Y$, then any morphism $\psi : X \to Y$ induces a map $\hat{\psi} : C_X \to C_Y$ with the property that

$$\psi\left(\overline{X_c}\right) \subseteq \overline{Y_{\hat{\psi}(c)}} \tag{1.1}$$

for all $c \in C_X$. Hence the structure of irreducible components, as described above, is fundamental to questions about the possibility of mapping one sofic shift space into or onto another. Note that even when X and Y are both SFTs and Y is not irreducible, this property of a morphism is still non-trivial, albeit well known. When Y is both SFT and irreducible, it is trivial even when X is strictly sofic.

The irreducible components impose a structure on the periodic orbits which must also be respected by maps between sofic shift spaces. As pointed out by Boyle, the problem about embedding and factorization among sofic shift spaces has to do with certain 'congruence conditions on allowed periodic block counts'; cf. p. 550 of [B]. These congruence conditions reflect how the periodic orbits relate to the irreducible components. Although a periodic point is contained in one and only one irreducible component, it can easily be contained in more than one of the *closures* of the irreducible components, and this leads to a notion, which we call affiliation of a periodic orbit to an irreducible component. Specifically, given a minimal cycle q in X, where 'cycle' means a word with the property that q^∞ is an element of X, we say that the periodic point q^∞ is *d-affiliated to an irreducible component* X_c when there are words u, v occurring in X_c, both of which are synchronizing for $R(X)$, such that $uq^{id}v$ is a word in $R(X)$ for all $i \in \mathbb{N}$. Here d can be any natural number. It turns out that q^∞ is 1-affiliated to X_c if and only if there is an irreducible SFT in X which contains q^∞ and allows words from X_c that are synchronizing for $R(X)$. 1-affiliation is closely related to the notion of receptive periodic points introduced by Boyle in [B]. In fact, in an irreducible sofic shift space a periodic point is receptive if and only if it is 1-affiliated to the unique irreducible component which is dense in X, and which we call *the top component* in X. It turns out that the way the periodic points affiliate to the irreducible components is a conjugacy invariant, and that arbitrary maps between sofic shift spaces must respect this structure in a way which resembles the way in which they must respect the periods. To make this precise, we say that a periodic point $x \in \operatorname{Per} X$ has *affiliation number* k to the irreducible component X_c when period(x) divides k and x is $\frac{k}{\operatorname{period}(x)}$-affiliated to X_c. It then turns out that any morphism $\psi : X \to Y$ must respect the affiliation numbers of the periodic points in the sense that

$$\begin{gathered} x \text{ has affiliation number } k \text{ to } X_c \\ \Downarrow \\ \psi(x) \text{ has affiliation number } k \text{ to } Y_{\hat{\psi}(c)}, \end{gathered}$$

where $X_c \to Y_{\hat{\psi}(c)}$ is the map of irreducible components induced by ψ. In the case where X and Y are SFTs this property of ψ reduces to the obvious one, that period$(\psi(x))$ divides period(x). In general the property is much more complex and subtle, and it explains why conditions on entropy and periods are not enough to ensure the existence of maps between (mixing) sofic shift spaces. We will refer to this property by saying that 'ψ preserves affiliation numbers'.

The main results of the paper say that for a large class of mixing sofic shift spaces, the condition that a morphism must preserve the affiliation numbers to top components gives rise to the appropriate substitutes for the conditions on periods in Krieger's embedding theorem and Boyle's factor theorem. In order to formulate the results so that they can be easily compared to the theorems of Krieger and Boyle, we introduce the following notation. Let X and Y be irreducible sofic shift spaces, with top components X_c and Y_c. We write $\operatorname{Per} X \hookrightarrow \operatorname{Per} Y$ when there is a period-preserving injective map $\lambda : \operatorname{Per} X \to \operatorname{Per} Y$ from the periodic points in X into the periodic points in Y such that $\lambda(x)$ is d-affiliated to X_c when x is d-affiliated to Y_c. We write $\operatorname{Per} X \searrow \operatorname{Per} Y$ when there is a map $\lambda : \operatorname{Per} X \to \operatorname{Per} Y$ such that $\operatorname{period}(\lambda(x))$ divides $\operatorname{period}(x)$ and $\lambda(x)$ is $\frac{\operatorname{period}(x)d}{\operatorname{period}(\lambda(x))}$-affiliated to Y_c when x is d-affiliated to X_c. The main results can now be formulated as follows:

Theorem 1.1. *Let X and Y be sofic shift spaces, X mixing with transparent affiliation pattern. If X embeds into Y, there is an irreducible component Y_c in Y such that $X \subseteq \overline{Y_c}$,*

a) $\operatorname{Per} X \hookrightarrow \operatorname{Per} \overline{Y_c}$, *and*
b) $h(X) \leq h\left(\overline{Y_c}\right)$.

Conversely, if there is an irreducible component Y_c in Y such that a) *holds and*

b′) $h(X) < h\left(\overline{Y_c}\right)$,

then $X \subseteq \overline{Y_c} \subseteq Y$.

Theorem 1.2. *Let X and Y be sofic shift spaces, X irreducible with transparent affiliation pattern and Y mixing. Assume that $h(X) > h(Y)$. Then Y is a factor of X if and only if* $\operatorname{Per} X \searrow \operatorname{Per} Y$.

We refrain here from giving the precise definition of what it means for an irreducible sofic shift space to have transparent affiliation pattern. See Section 9. It suffices to say that all mixing sofic shift spaces which are inclusive, as defined by Boyle in [B], and all near Markov shifts, as defined by Boyle and Krieger in [BK], have transparent affiliation pattern, and that there are many others; but that not all mixing sofic shift spaces have this property. In fact, we give examples to show that neither of the two theorems above holds true when X is allowed to be a general mixing sofic shift. The reason is that in the definition of $\hookrightarrow$ and $\searrow$, we consider only the affiliation pattern of the periodic points to the top component. The counterexamples we exhibit show that sufficient conditions for the existence of embeddings and factor maps between mixing sofic shift spaces must, in the general case, reflect the affiliation pattern of the periodic points to more than one irreducible component; presumably it will be necessary to consider all components in the general case. Furthermore, the irreducible components may have a global periodicity structure, i.e. be non-mixing, although the sofic shift as a whole *is* mixing, and this must also be taken into account in the general case. Example 9.14 illustrates this point. Not only will the right conditions be more complicated, but the proof of the sufficiency of such conditions must be considerably more complicated than the proofs we give here of Theorem 1.1 and Theorem 1.2. These proofs are mild elaborations of the techniques developed by Krieger and Boyle.

In neither of the two theorems above can we relax the mixing condition by replacing it with irreducibility. But if one of the shift spaces is an SFT, we can obtain versions of Theorem 1.1 and Theorem 1.2 in the irreducible case. Section 8 is devoted to such results. In particular, we are able to give necessary and sufficient conditions for an irreducible sofic shift space to be a factor of a given irreducible SFT of larger entropy.

The structure of derived subshifts and irreducible components is present and signifigant in a larger class of subshifts than the sofic subshifts; in fact, the notions are meaningful as soon as there are synchronizing words. For this reason we develop a substantial part of the theory for such subshifts, and illustrate the difference between the sofic case and the more general case of synchronized systems through examples.

2. Basic notation and terminology

Let $\mathcal{A}$ be a finite set, and let $\mathcal{A}^{\mathbb{Z}}$ denote the set of bi-infinite sequences

$$x = (x_i)_{i\in\mathbb{Z}} = \dots x_{-3}x_{-2}x_{-1}x_0x_1x_2\dots,$$

where $x_i \in \mathcal{A}$ for all i. When we equip $\mathcal{A}$ with its unique Hausdorff topology, we can consider $\mathcal{A}^{\mathbb{Z}}$ as a compact metric space in the product topology. An element $w = y_1y_2\dots y_k \in \mathcal{A}^k$ will be called a *word of length* k, and when $x \in \mathcal{A}^{\mathbb{Z}}$ we say that w *occurs in* x, and write $w \subseteq x$, when $w = x_{i+1}x_{i+2}\dots x_{i+k}$ for some $i \in \mathbb{Z}$. Similarly, when w, u are words we write $w \subseteq u$ when $u = x_1x_2\dots x_k$ and $w = x_ix_{i+1}x_{i+2}\dots x_j$ for some $i \leq j$ in $\{1,2,\dots,k\}$. The length of a word w will be denoted by $|w|$. The set of all words, including the empty word, will be denoted by $\mathbb{W}(\mathcal{A}^{\mathbb{Z}})$. Words in $\mathbb{W}(\mathcal{A}^{\mathbb{Z}})$ can be concatenated in the obvious way: When $w = x_1x_2\dots x_k, u = y_1y_2\dots y_n$ are words, then $wu = x_1x_2\dots x_ky_1y_2\dots y_n$ is also a word. To simplify notation we write w^2 instead of ww and more generally w^n for the word $ww\dots w$ obtained from n concatenations of w with itself. In fact, when the obvious ambiguity is irrelevant, we shall go even further and use self-explanatory expressions such as w^∞ for an element of $\mathcal{A}^{\mathbb{Z}}$ constructed by repeating w infinitely many times both to the right and to the left. When $x = (x_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ and $i < j$, we denote the word $x_ix_{i+1}\dots x_{j-1}$ by $x_{[i,j[}$. Thus $|x_{[i,j[}| = j - i$. The shift σ acts continuously on $\mathcal{A}^{\mathbb{Z}}$ in the standard way: $\sigma(x)_i = x_{i+1}$. A *subshift* X is a shift-invariant subset of $\mathcal{A}^{\mathbb{Z}}$, i.e. $X \subseteq \mathcal{A}^{\mathbb{Z}}$ and $\sigma(X) = X$. The set of words that occur in X, i.e. $\{w \in \mathbb{W}(\mathcal{A}^{\mathbb{Z}}) : w \subseteq x \text{ for some } x \in X\}$, will be denoted by $\mathbb{W}(X)$ and called *the language of* X. The set of words in $\mathbb{W}(X)$ of length n will be denoted by $\mathbb{W}_n(X)$. A subshift X is a *shift space* when X is closed in $\mathcal{A}^{\mathbb{Z}}$. For any subshift X the closure $\overline{X}$ of X in $\mathcal{A}^{\mathbb{Z}}$ is a shift space with the same language as X, i.e. $\mathbb{W}(\overline{X}) = \mathbb{W}(X)$. The structure of the set $\operatorname{Per} X$ of periodic points of a subshift X is a central theme in this paper. A period of a periodic point $x \in \operatorname{Per} X$ is a natural number $n \in \mathbb{N}$ such that $\sigma^n(x) = x$. The minimal period of a periodic point $x \in \operatorname{Per} X$ will be denoted by $\operatorname{period}(x)$, and we set

$$Q_n(X) = \{x \in \operatorname{Per} X : \operatorname{period}(x) = n\},$$

and $q_n(X) = \#Q_n(X)$. The *period*, $\operatorname{period}(X)$, of X is by definition the greatest common divisor of $\{\operatorname{period}(x) : x \in \operatorname{Per} X\}$. For each $j \in \mathbb{N}$, we set

$$P_j(X) = \#\{x \in X : \sigma^j(x) = x\}.$$

A *cycle* w in $\mathbb{W}(X)$ is a word w in $\mathbb{W}(X)$ such that $w^\infty \in X$. A *minimal cycle* in $\mathbb{W}(X)$ is a cycle $w \in \mathbb{W}(X)$ such that $|w| = \operatorname{period}(w^\infty)$. A subshift X is irreducible when the following holds: For any pair $w, v \in \mathbb{W}(X)$, there is a word $a \in \mathbb{W}(X)$ such that $wav \in \mathbb{W}(X)$. X is *mixing* when there is an N, possibly depending on w, v, such that for all $n \geq N$ there is a word $a \in \mathbb{W}(X)$ of length n such that $wav \in \mathbb{W}(X)$. If there is such an N which works for all words, we call N a *transition length* for X. Any shift space X can be described by the list $\mathcal{F}$ of words in $\mathbb{W}(\mathcal{A}^{\mathbb{Z}})$ which it disallows in the sense that

$$X = \{x \in \mathcal{A}^{\mathbb{Z}} : x_{[i,j[} \notin \mathcal{F} \ \forall i, j, \ i < j\}.$$

A shift space is a *shift of finite type*, abbreviated SFT, when the list $\mathcal{F}$ of forbidden words can be chosen to be finite. SFTs are often presented by graphs, either as vertex shifts or edge shifts; cf., e.g., [LM]. Given a graph G, we denote by X_G the corresponding edge-shift. A word $w \in \mathbb{W}(X)$ is *synchronizing for X* when

$$u, v \in \mathbb{W}(X), \ uw, wv \in \mathbb{W}(X) \ \Rightarrow \ uwv \in \mathbb{W}(X).$$

Note that when $u, w \in \mathbb{W}(X), w \subseteq u$, and w is synchronizing for X, then u is also synchronizing for X. In an SFT there is an $M \in \mathbb{N}$, which we call *a step-length*, such that all allowed words of length M or more are synchronizing. In fact, the SFTs are characterized as the shift spaces with this property; cf. Theorem 2.1.8 of [LM]. Following [BH], we call an irreducible shift space with a synchronizing word a *synchronized system*. An irreducible sofic shift space is an example of a synchronized system. A sofic shift is often conveniently described by a labeled graph $(G, \mathcal{L})$; cf. [LM]. When X is an irreducible sofic shift space, there is a canonical labeled graph with certain minimality properties which presents X, and we shall refer to the corresponding factor map $X_G \to X$ as the *Fischer cover* of X. See §3.3 of [LM]. A map $X \to Y$ between shift spaces will be called a *morphism* when it is continuous and commutes with the shift. An *embedding* is then an injective morphism and a *factor map* is a surjective morphism of shift spaces.

We refer to [LM] for any notion from the theory of shift spaces which we do not explicitly define.

3. Irreducible components in shift spaces with synchronizing words

Let X be a shift space. Set $R(X) = \overline{\operatorname{Per} X}$. Let $\mathbb{S}(X)$ denote the set of synchronizing words for $R(X)$. For $s, t \in \mathbb{S}(X)$ we write $s \sim t$ when there are words $x, y \in \mathbb{W}(R(X))$ such that $sxt, tys \in \mathbb{W}(R(X))$. Then $\sim$ is an equivalence relation in $\mathbb{S}(X)$. Note that $s \sim t$ if and only if there is an $x \in R(X)$ such that $s \subseteq x$ and $t \subseteq x$. When $w \in \mathbb{S}(X)$, we let $[w]$ denote its equivalence class in $\mathbb{S}(X)/\sim$, and we denote the set of equivalence classes, $\mathbb{S}(X)/\sim$, by $\mathcal{S}(X)$.

Consider an element $\alpha \in \mathcal{S}(X)$. Let $X_{(\alpha,0)}$ denote the set of elements $x \in R(X)$ for which

$$\sup_{i \in \mathbb{Z}} \left(\inf\{j \geq i : \ \exists w \in \alpha, \ w \subseteq x_{[i,j[}\} \right)$$

is finite. We use here the convention that $\inf \emptyset = +\infty$. Thus $X_{(\alpha,0)}$ consists of the elements of $R(X)$ with an upper bound on the gaps between occurrences of words from α. $X_{(\alpha,0)}$ is clearly a subshift of $R(X)$, but generally not closed. It is easily seen that $X_{(\alpha,0)}$ is irreducible and that $\overline{X_{(\alpha,0)}}$ is a synchronized system. For any synchronized system Y, we define *the synchronized entropy* $h_{syn}(Y)$ to be

$$h_{syn}(Y) = \limsup_{n} \frac{1}{n} \log \#\{a \in \mathbb{W}_n(Y) : \ mam \in \mathbb{W}(Y)\},$$

where m is an arbitrary synchronizing word in $\mathbb{W}(Y)$. We use here the convention that $\log 0 = 0$. If m' is another synchronizing word in $\mathbb{W}(Y)$, we can find words

$u, v \in \mathbb{W}(Y)$ such that $m'um, mvm' \in \mathbb{W}(Y)$. Since m and m' are synchronizing, we find that $mam \in \mathbb{W}(Y) \Rightarrow m'umamvm' \in \mathbb{W}(Y)$, and it follows that

$$\#\{a \in \mathbb{W}_n(Y) : mam \in \mathbb{W}(Y)\} \leq \#\{b \in \mathbb{W}_{n+|u|+|v|+2|m|}(Y) : m'bm' \in \mathbb{W}(Y)\}.$$

Hence

$$\begin{aligned} &\limsup_n \frac{1}{n} \log \#\{a \in \mathbb{W}_n(Y) : \ mam \in \mathbb{W}(Y)\} \\ &\qquad \leq \limsup_n \frac{1}{n} \log \#\{a \in \mathbb{W}_n(Y) : \ m'am' \in \mathbb{W}(Y)\}, \end{aligned}$$

proving that the definition of $h_{syn}(Y)$ is independent of the choice of synchronizing word m. Note that

$$\{a \in \mathbb{W}_n(Y) : \ mam \in \mathbb{W}(Y)\} = \{a \in \mathbb{W}_n(Y) : (am)^\infty \in Y\}.$$

Thus $\#\{a \in \mathbb{W}_n(Y) : \ mam \in \mathbb{W}(Y)\} \leq P_{n+|m|}(Y)$ for all n, and hence $h_{syn}(Y) \leq \limsup_{n\to\infty} \frac{1}{n} \log P_n(Y)$. It follows that

$$h_{syn}(Y) \leq h(Y); \tag{3.1}$$

cf. Proposition 4.1.15 of [LM]. There are examples of synchronized systems Y for which $h_{syn}(Y) < h(Y)$; cf. Example 3.2 of [P].

Lemma 3.1. *Let Y be an irreducible sofic shift space. Then $h_{syn}(Y) = h(Y)$.*

Proof. Consider the Fischer cover $\pi : Y_G \to Y$ of Y, and consider a synchronizing word $m \in \mathbb{W}(Y)$. Since m is magic for π by (e.g.) Lemma 1.1 of [T], all elements of $\pi^{-1}(m)$ have the same terminal vertex, say i. Let $v \in \pi^{-1}(m)$. Since Y_G is irreducible, there is a path u in G with initial vertex i such that $uv \in \mathbb{W}(Y_G)$. Then $m' = \pi(uv)$ is synchronizing for Y. Let L_n denote the set of loops in Y_G of length n that start and end at i. Since π is right-resolving,

$$\pi : L_n \to \{a \in \mathbb{W}_n(Y) : m'am' \in \mathbb{W}(Y)\}$$

is injective. Hence

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n} \log \#L_n \\ &\quad \leq \limsup_{n\to\infty} \frac{1}{n} \log \#\{a \in \mathbb{W}_n(Y) : m'am' \in \mathbb{W}(Y)\} = h_{syn}(Y). \end{aligned} \tag{3.2}$$

It is well known that $\limsup_{n\to\infty} \frac{1}{n} \log \#L_n = h(Y_G)$ (cf., e.g., Exercise 4.5.14 of [LM]). Since $h(Y_G) = h(Y)$, (3.2) tells us that $h(Y) \leq h_{syn}(Y)$. □

We now return to the general shift space X. We can associate to $X_{(\alpha,0)}$ an irreducible labeled graph Γ_α in the following way (cf. [FF] or [LM], §13.5): For each $m \in \alpha$, let $\mathcal{F}(m) = \{x \in \mathbb{W}\left(X_{(\alpha,0)}\right) : mx \in \mathbb{W}\left(X_{(\alpha,0)}\right)\}$. The vertices of Γ_α consist of $\{\mathcal{F}(m) : m \in \alpha\}$, and there is an edge labeled a from $\mathcal{F}(m)$ to $\mathcal{F}(m')$ when $a \in \mathcal{F}(m)$ and $\mathcal{F}(ma) = \mathcal{F}(m')$. Γ_α is a cover of $\overline{X_{(\alpha,0)}}$ in the sense of [FF]. When T denotes the adjacency matrix of Γ_α,

$$h(\Gamma_\alpha) = \limsup_n \frac{1}{n} \log (T^n)_{ii}$$

is independent of i, and we call it the *Gurevič entropy* of Γ_α; cf. [Gu], [P]. The following result is a fusion of a result of Marcus, Proposition 3 of [M1], and an unpublished result of Krieger.

Theorem 3.2. *There is a sequence $A_{\alpha,1} \subseteq A_{\alpha,2} \subseteq A_{\alpha,3} \subseteq \dots$ of irreducible SFTs in X such that*

$$X_{(\alpha,0)} = \bigcup_{n=1}^{\infty} A_{\alpha,n},$$

and

$$\lim_{n\to\infty} h\left(A_{\alpha,n}\right) = \sup\{h(A) : A \subseteq X_{(\alpha,0)} \text{ is an irreducible SFT}\}$$
$$= h\left(\Gamma_\alpha\right) = h_{syn}\left(\overline{X_{(\alpha,0)}}\right)$$
$$= \lim_{n\to\infty} \frac{1}{n \operatorname{period}\left(X_{(\alpha,0)}\right)} \log \#\{a \in \mathbb{W}_{n \operatorname{period}\left(X_{(\alpha,0)}\right)}(X_{(\alpha,0)}) : mam \in \mathbb{W}\left(X_{(\alpha,0)}\right)\}$$

for all $m \in \alpha$ with the property that $m^2 \in \mathbb{W}\left(X_{(\alpha,0)}\right)$. When $R(X)$ is mixing we can choose the $A_{\alpha,n}$'s to be mixing.

For the proof of Theorem 3.2 we first define a graph for each $n \in \mathbb{N}$ as follows: The vertex-set is the set $\mathbb{W}_n(R(X))$ of words in $\mathbb{W}(R(X))$ of length n. There is an oriented edge from $b_1b_2\dots b_n \in \mathbb{W}_n(R(X))$ to $d_1d_2\dots d_n \in \mathbb{W}_n(R(X))$ if and only if $b_2b_3\dots b_n = d_1d_2\dots d_{n-1}$ and there is a word $w \in \alpha$ such that $w \subseteq b_2b_3\dots b_n$. Let $B_{\alpha,n}$ be the corresponding vertex shift, and note that there is an obvious embedding $B_{\alpha,n} \to R(X)$ whose image is

$$\{x \in R(X) : x_{[i,i+n-1[} \text{ contains an element of } \alpha \text{ for all } i \in \mathbb{Z}\}.$$

We identify $B_{\alpha,n}$ with its image in $R(X)$. Clearly, $\bigcup_{n=1}^{\infty} B_{\alpha,n} = X_{(\alpha,0)}$. Note that $B_{\alpha,n} \subseteq B_{\alpha,n+1}$. The $B_{\alpha,n}$'s need not be irreducible, so we have to exchange them with other SFTs that are. We shall need the following lemma.

Lemma 3.3. *Let $w,u \in \mathbb{W}(X_{(\alpha,0)}) \cap \mathbb{S}(X)$. Then $[w] = [u] = \alpha$.*

Proof. There is an $n \in \mathbb{N}$ such that $w,u \in \mathbb{W}(B_{\alpha,n})$. By definition of $B_{\alpha,n}$, this implies that there are words $x,y \in \mathbb{W}(R(X))$ and words $a,b,c,d \in \alpha$ such that $w \subseteq axb$ and $u \subseteq cyd$. Since $b \sim c$, there is a word $v \in \mathbb{W}(R(X))$ such that $bvc \in \mathbb{W}(R(X))$. By using that b and c are synchronizing for $R(X)$ we find that $axbvcyd \in \mathbb{W}(R(X))$. We can therefore choose a subword z of $axbvcyd$, containing v, such that $wzu \in \mathbb{W}(R(X))$. By symmetry there is also a word $r \in \mathbb{W}(R(X))$ such that $urw \in \mathbb{W}(R(X))$. Hence $[w] = [u]$. Since $w \subseteq axb \subseteq axbvcyd$ and $u \subseteq cyd \subseteq axbvcyd$, it follows that $[w] = [u] = \alpha$. □

Completing the proof of Theorem 3.2. We claim that for each pair $n,k \in \mathbb{N}$, there are an $m_n \in \mathbb{N}, m_n \geq k$, and an irreducible component V_n of B_{α,m_n} such that $B_{\alpha,n} \subseteq V_n$. To see this, choose a periodic point q_i in each of the irreducible components of $B_{\alpha,n}$. Since each irreducible component of $B_{\alpha,n}$ is contained in an irreducible component of $B_{\alpha,m}$ when $m > n$, it suffices to find $m_n \geq \max\{n,k\}$ such that all the q_i are in the same irreducible component of B_{α,m_n}. To this end it suffices to consider a pair p,q of periodic points in $B_{\alpha,n}$, and find $m \geq \max\{n,k\}$ such that p and q are in the same irreducible component of $B_{\alpha,m}$. Since p,q are periodic, there are cycles $c,d \in \mathbb{W}(R(X))$ such that $p = c^\infty, q = d^\infty$. Since $p,q \in B_{\alpha,n}$, c^n and d^n must be synchronizing for $R(X)$, i.e., $c^n, d^n \in \mathbb{S}(X)$. By Lemma 3.3 there are words $x,y \in \mathbb{W}(R(X))$ such that $c^nxd^n, d^nyc^n \in \mathbb{W}(R(X))$ and $[c^n] = [d^n] = \alpha$. Since c^n and d^n are both synchronizing for $R(X)$, $c^\infty xd^\infty, d^\infty yc^\infty \in R(X)$. Now choose $m > \max\{k, n|c| + n|d| + |x| + |y|\} + 1$. Since every word of length $m-1$ in $c^\infty xd^\infty$ or $d^\infty yc^\infty$ contains either c^n or d^n, both of which are elements in α, we see that $r^1 = c^\infty x d^\infty$ and $r^2 = d^\infty y c^\infty$ are both elements of $B_{\alpha,m}$. Note that $\lim_{k\to\infty} \operatorname{dist}(\sigma^k(r_1), O(q)) = \lim_{k\to-\infty} \operatorname{dist}(\sigma^k(r_1), O(p)) = \lim_{k\to\infty} \operatorname{dist}(\sigma^k(r_2), O(p)) = \lim_{k\to-\infty} \operatorname{dist}(\sigma^k(r_2), O(q)) = 0$, where $O(p)$ and $O(q)$ are the orbits of p and q, respectively. It follows that p and q must be in the same irreducible component of $B_{\alpha,m}$. We have now established the claim and can easily complete the proof by inductively choosing $m_n \in \mathbb{N}$ such that $m_{n-1} < m_n$, and take $A_{\alpha,n}$ to be the irreducible component of B_{α,m_n} containing $B_{\alpha,m_{n-1}}$.

Assume that $R(X)$ is mixing and consider $w \in \alpha$. There are then an $N \in \mathbb{N}$ and for each $n \geq N$ a word $a_n \in \mathbb{W}_n(R(X))$ such that $w a_n w \in \mathbb{W}(R(X))$. Then $(w a_n)^\infty \in X_{(\alpha,0)}$ is a periodic point of period $|w| + n$. There are therefore periodic points $p, q \in X_{(\alpha,0)}$ whose minimal periods are mutually prime. Since $p, q \in A_{(\alpha,k)}$ for all large enough k, it follows that $A_{(\alpha,k)}$ is aperiodic and hence mixing for all large enough k.

To prove the identities involving entropy, let $\epsilon > 0$ and $n \in \mathbb{N}$ be arbitrary. Set $t = \sup\{h(A) : A \subseteq X_{(\alpha,0)} \text{ is an irreducible SFT}\} \in [0, \infty]$. Let $A \subseteq X_{(\alpha,0)}$ be an irreducible SFT such that

$$h(A) \geq \min\{t, n\} - \epsilon. \tag{3.3}$$

Since $A \subseteq X_{(\alpha,0)}$, there is a word $u \in \mathbb{W}(A) \cap \alpha$ which is synchronizing for A. Since A is irreducible,

$$h(A) = \limsup_{k\to\infty} \frac{1}{k} \log \#\{a \in \mathbb{W}_k(A) : (au)^\infty \in A\}; \tag{3.4}$$

cf. Lemma 3.1. In particular,

$$h_{syn}\left(\overline{X_{(\alpha,0)}}\right) \geq h(A). \tag{3.5}$$

Combining (3.3) and (3.4), we deduce that there is an $N \in \mathbb{N}$ so large that

$$\frac{1}{N + |u|} \log \#\{a \in \mathbb{W}_N(A) : (au)^\infty \in A\} \geq \min\{t, n\} - 2\epsilon. \tag{3.6}$$

For any finite set of elements $a_1, a_2, \ldots, a_k \in \{a \in \mathbb{W}_N(A) : (au)^\infty \in A\}$,

$$(a_1 u a_2 u a_3 \ldots a_k u)^\infty \in B_{\alpha, N+|u|+1},$$

because u is synchronizing for $R(X)$. By construction there is an $A_{\alpha,l}$ such that $B_{\alpha,N+|u|+1} \subseteq A_{\alpha,l}$. It follows that

$$P_{kN+k|u|}\left(A_{\alpha,l}\right) \geq \left(\#\{a \in \mathbb{W}_N(A) : (au)^\infty \in A\}\right)^k$$

for all $k \in \mathbb{N}$. Hence

$$\begin{aligned} h\left(A_{\alpha,l}\right) &= \limsup_{j\to\infty} \frac{1}{j} \log P_j\left(A_{\alpha,l}\right) \\ &\geq \limsup_{k\to\infty} \frac{1}{kN + k|u|} \log P_{kN+k|u|}\left(A_{\alpha,l}\right) \\ &\geq \limsup_{k\to\infty} \frac{k}{kN + k|u|} \log \#\{a \in \mathbb{W}_N(A) : (au)^\infty \in A\} \\ &= \frac{1}{N + |u|} \log \#\{a \in \mathbb{W}_N(A) : (au)^\infty \in A\}. \end{aligned}$$

Combined with (3.6), this yields the conclusion that $\lim_{n\to\infty} h\left(A_{\alpha,n}\right) = t$. From (3.5) we deduce that $t \leq h_{syn}\left(\overline{X_{(\alpha,0)}}\right)$.

Choose a path r in Γ_α which is labeled u, and let v be the terminal vertex of r. By definition of Γ_α, any path whose label terminates with u must have v as terminal vertex (i.e. u is 'magic'). Hence

$$\#\{a \in \mathbb{W}_k\left(X_{(\alpha,0)}\right) : uau \in \mathbb{W}\left(X_{(\alpha,0)}\right)\} \leq N_{k+|u|},$$

where N_i is the number of loops of length i in Γ_α that are based at v. Since $\#\{a \in \mathbb{W}_k(A) : (au)^\infty \in A\} \leq \#\{a \in \mathbb{W}_k\left(X_{(\alpha,0)}\right) : uau \in \mathbb{W}\left(X_{(\alpha,0)}\right)\}$, we can combine this estimate with (3.3)-(3.5) to conclude that

$$\min\{t, n\} - \epsilon \leq h_{syn}\left(\overline{X_{(\alpha,0)}}\right) \leq \limsup_k \frac{1}{k} \log N_k.$$

Since $\limsup_k \frac{1}{k} \log N_k = h(\Gamma_\alpha)$, we conclude that $t \leq h_{syn}\left(\overline{X_{(\alpha,0)}}\right) \leq h(\Gamma_\alpha)$. Consider a finite irreducible subgraph $H \subseteq \Gamma_\alpha$ containing r. Let X_H be the corresponding SFT and let $\pi : X_H \to R(X)$ be the map obtained by reading the labels in H. Since Γ_α is right-resolving, $h(X_H) = h(S)$, where $S = \pi(X_H) \subseteq R(X)$. Note that $u \in \mathbb{W}(S)$, since H contains r. Since S is an irreducible sofic shift space, (3.4) also holds for S; cf. Lemma 3.1. We can therefore repeat the argument from above to show that for any $\epsilon > 0$, there is an $l \in \mathbb{N}$ so large that $h(A_{\alpha,l}) \geq h(S) - 2\epsilon$. Hence $\lim_{n\to\infty} h(A_{\alpha,n}) \geq h(S) = h(X_H)$, and we conclude from [Gu] that $\lim_{n\to\infty} h(A_{\alpha,n}) \geq h(\Gamma_\alpha)$.

We have now established the first three identities. To establish the last, observe that $m^\infty \in A_{\alpha,l}$ and $\mathrm{period}(A_{\alpha,l}) = \mathrm{period}(X_{(\alpha,0)})$ for all l large enough. It follows that

$$\#\{a \in \mathbb{W}_{n\,\mathrm{period}\left(X_{(\alpha,0)}\right)}\left(X_{(\alpha,0)}\right) : mam \in \mathbb{W}\left(X_{(\alpha,0)}\right)\} \neq 0$$

for all $n \in \mathbb{N}$ large enough. By using that $m^2 \in \mathbb{W}\left(X_{(\alpha,0)}\right)$ we find that

$$\begin{aligned}
&\#\{a \in \mathbb{W}_{n+k+\mathrm{period}\left(X_{(\alpha,0)}\right)|m|}(X_{(\alpha,0)}) : mam \in \mathbb{W}\left(X_{(\alpha,0)}\right)\} \\
&\geq \#\{a \in \mathbb{W}_n(X_{(\alpha,0)}) : mam \in \mathbb{W}\left(X_{(\alpha,0)}\right)\} \\
&\quad \times \#\{a \in \mathbb{W}_k(X_{(\alpha,0)}) : mam \in \mathbb{W}\left(X_{(\alpha,0)}\right)\}
\end{aligned}$$

for all n, k. It follows that

$$\lim_{n\to\infty} \frac{1}{n\,\mathrm{period}\left(X_{(\alpha,0)}\right)} \log \#\{a \in \mathbb{W}_{n\,\mathrm{period}\left(X_{(\alpha,0)}\right)}\left(X_{(\alpha,0)}\right) : mam \in \mathbb{W}\left(X_{(\alpha,0)}\right)\}$$

exists. This limit is clearly dominated by $h_{syn}\left(\overline{X_{(\alpha,0)}}\right)$. On the other hand, it dominates $h(A_{\alpha,l})$ when $\mathrm{period}(A_{\alpha,l}) = \mathrm{period}(X_{(\alpha,0)})$ and $m^\infty \in A_{\alpha,l}$. The last identity follows from this. □

The subshifts $X_{(\alpha,0)}, \alpha \in \mathcal{S}(X)$, will be called *the irreducible components at level* 0 in X. They are mutually disjoint:

Lemma 3.4. *Let $X_{(\alpha',0)} \neq X_{(\alpha,0)}$ be two different irreducible components at level* 0 *in X. Then $\overline{X_{(\alpha,0)}} \cap X_{(\alpha',0)} = \emptyset$.*

Proof. Assume that there is an element x in the intersection $\overline{X_{(\alpha,0)}} \cap X_{(\alpha',0)}$. Then $x \in A_{\alpha',n}$ for some n, and hence $x_{[0,n[}$ must contain an element $w \in \alpha'$. On the other hand, because $x \in \overline{X_{(\alpha,0)}}$, there are an $m \in \mathbb{N}$ and an element $y \in A_{\alpha,m}$ such that $y_{[0,n[} = x_{[0,n[}$. By definition of $A_{\alpha,m}$, this implies that there are elements $u, v \in \alpha$ and words $a, b \in \mathbb{W}(R(X))$ such that $uax_{[0,n[}bv \in \mathbb{W}(R(X))$. Since $w \subseteq x_{[0,n[}$, this implies that $w \sim u$, i.e. $\alpha = \alpha'$. This contradicts that $X_{(\alpha',0)} \neq X_{(\alpha,0)}$. □

Let $Z \subseteq X$ be a subshift of X. It is easy to see that Z is the closure of an irreducible component at level 0 in X if and only if Z is a maximal subshift of $R(X)$ with the following two properties:

a) Z is transitive in the sense that there is a point $z \in Z$ whose orbit is dense in Z, and
b) $\mathbb{W}(Z)$ contains a word which is synchronizing for $R(X)$.

Lemma 3.5. *Let X be a synchronized system. Then there is exactly one element, α_0, in $\mathcal{S}(X)$, and $X_{(\alpha_0,0)}$ is dense in X.*

Proof. Let w be a synchronizing word in $\mathbb{W}(X)$, and let $a \in \mathbb{W}(X)$ be arbitrary. Since X is irreducible, there are words $b_1, b_2 \in \mathbb{W}(X)$ such that $wb_1ab_2w \in \mathbb{W}(X)$. By using that w is synchronizing it follows that $(wb_1ab_2)^\infty \in R(X)$. Hence $a \in \mathbb{W}(R(X))$, and we conclude that $X = R(X)$. Since $R(X)$ is irreducible, all elements of $\mathbb{S}(X)$ are equivalent, i.e. $\mathcal{S}(X)$ contains exactly one element, $\alpha_0 = [w]$. To see that $X_{(\alpha_0,0)}$ is dense, note that the periodic element $(wb_1ab_2)^\infty$ is in $X_{(\alpha_0,0)}$. Since a was an arbitrary word in X, we see that $\mathbb{W}(X_{(\alpha_0,0)}) = \mathbb{W}(X)$. □

By Lemma 3.5 there is exactly one irreducible component at level 0 in a synchronized system, and we call it *the top component*. The first lemma in the following subsection shows that it is the period of this component which determines whether a synchronized system is mixing or not.

3.1. **Global periodic structure in a synchronized system.**

Lemma 3.6. *Let X be a synchronizing system with top component X_c. Then X is mixing if and only if* $\text{period}(X_c) = 1$.

Proof. If X is mixing, the $A_{\alpha,n}$'s of Theorem 3.2 can be taken to be mixing. As is well known, this implies that $\text{period}(A_{\alpha,n}) = 1$; cf., e.g., Proposition 4.5.10 (4) of [LM]. Since $A_{\alpha,n} \subseteq X_c$, we conclude that $\text{period}(X_c) = 1$.

Conversely, if $\text{period}(X_c) = 1$, there is a finite set $p_1, p_2, \ldots, p_N$ of periodic points in X_c such that the greatest common divisor of $\text{period}(p_1), \ldots, \text{period}(p_N)$ is 1. Since $\{p_1, \ldots, p_N\} \subseteq A_{\alpha,l}$ for all large enough l, we conclude that $\text{period}(A_{\alpha,l}) = 1$, and hence that $A_{\alpha,l}$ is mixing for all large l. Since

$$\mathbb{W}(X) = \mathbb{W}(X_c) = \bigcup_l \mathbb{W}(A_{\alpha,l}),$$

we deduce that X is mixing. □

Remark 3.7. In general, $\text{period}(X_c)$ is larger than $\text{period}(X)$, even for irreducible sofic shift spaces. As an example, consider the irreducible sofic shift space Y presented by the labeled graph shown in Figure 1.

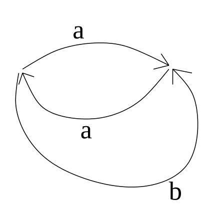


FIGURE 1.

Here period$(Y) = 1$ since Y has a fixed point. But the period of the top component in Y is 2, and Y is not mixing. Note that period(Y_c) is the same as the period of the graph in the Fischer cover of Y in this example. This is the case in general for irreducible sofic shift spaces; cf. Lemma 6.21.

Given a shift space X, we denote by X^n the nth power of X; cf. p. 14 in [LM]. The next lemma shows that under the passage to higher powers the structure of irreducible components (at level 0) behaves in much the same way as for an SFT.

Lemma 3.8. *Let $X_{(\alpha,0)}$ be an irreducible component at level 0 in the shift space X. Let $n \in \mathbb{N}$, and set $L = \{[w] \in \mathcal{S}(X^n) : w \in \alpha \cap \mathbb{W}(R(X)^n)\}$. Then $\#L < \infty$, $\#L$ divides both n and* period$(X_{(\alpha,0)})$*, and there are a partition*

$$X_{(\alpha,0)} = \bigsqcup_{\beta \in L} (X^n)_{(\beta,0)} \tag{3.7}$$

and a transitive permutation l of L such that $\sigma\left((X^n)_{(\beta,0)}\right) = (X^n)_{(l(\beta),0)}$ for all $\beta \in L$.

Proof. Note first that $\mathbb{S}(X^n) = \mathbb{S}(X) \cap \mathbb{W}(R(X)^n)$. If $u, v \in \mathbb{S}(X^n)$ are equivalent, they are also equivalent in $\mathbb{S}(X)$, so there is a well-defined map $d : \mathcal{S}(X^n) \to \mathcal{S}(X)$ such that $d[w] = [w]$ for $w \in \mathbb{S}(X^n)$. When $u \in \mathbb{S}(X)$, there is a word $\tilde{u} \in \mathbb{S}(X) \cap \mathbb{W}(R(X)^n) = \mathbb{S}(X^n)$ such that $u \subseteq \tilde{u}$. Since $d[\tilde{u}] = [u]$, this shows that d is surjective. Note that $L = d^{-1}(\alpha)$. It is clear that $(X^n)_{(\beta,0)} \subseteq X_{(\alpha,0)}$ for all $\beta \in L$. Let $x \in X_{(\alpha,0)}$. There is then a K such that $x_{[i,i+K[} \in \alpha$ for all $i \in \mathbb{Z}$. Set $k = [\frac{K}{n}] + 1$ and note that $x_{[ni,ni+nk[} \in \mathbb{S}(X) \cap \mathbb{W}(R(X)^n) = \mathbb{S}(X^n)$ for all $i \in \mathbb{Z}$. It follows that $x \in (X^n)_{(\beta,0)}$ for some $\beta \in \mathcal{S}(X^n)$. Since $d([x_{[ni,ni+nk[}]) = \alpha$ for all i, we conclude that $\beta \in L$. Hence $X_{(\alpha,0)} = \bigcup_{\beta \in L} (X^n)_{(\beta,0)}$. By Lemma 3.4 this union is a partition.

Let $x \in (X^n)_{(\beta,0)}, \beta \in L$. Then $x_{[1,1+nk[} \in \mathbb{W}(R(X)^n) \cap \alpha$ for all large enough $k \in \mathbb{N}$. Since $[y_{[1,1+nk[}] = [x_{[1,1+nm[}]$ in $\mathcal{S}(X^n)$ when $x, y \in (X^n)_{(\beta,0)}$ and k and m are large enough, it follows that we can define $l : L \to L$ such that $l(\beta) = \lim_{k\to\infty}[x_{[1,1+nk[}]$ when $x \in (X^n)_{(\beta,0)}$. Then $\sigma\left((X^n)_{(\beta,0)}\right) \subseteq (X^n)_{(l(\beta),0)}$ for all $\beta \in L$. Similarly, we can define an inverse r of l such that

$$r(\beta) = \lim_{k\to\infty}[x_{[-1,-1+nk[}]$$

when $x \in (X^n)_{(\beta,0)}$. Thus l is a permutation of L, as asserted. To see that l acts transitively on L, consider two elements $\beta, \gamma \in L$. Let $x \in (X^n)_{(\beta,0)}, y \in (X^n)_{(\gamma,0)}$. By Theorem 3.2 there is an irreducible SFT $A \subseteq X_{(\alpha,0)}$ containing both x and y. It follows that there are a word $v \in \mathbb{W}(R(X))$ and an element $z \in X_{(\alpha,0)}$ such that $z_i = x_i, i \leq 0$, $z_i = v_i, i = 1, 2, \ldots, |v|$, and $z_i = y_{i-|v|}, i > |v|$. Then $z \in (X^n)_{(\beta,0)}$. To see this, observe first that by (3.7) $z \in (X^n)_{(\delta,0)}$ for some $\delta \in L$. We can then choose $M \in \mathbb{N}$ so large that $x_{[ni,ni+Mn[} \in \beta$ and $z_{[ni,ni+Mn[} \in \delta$ for all $i \in \mathbb{Z}$. Since $z_i = x_i, i \leq 0$, this implies that $\beta = \delta$, as asserted. Similarly, it follows that $\sigma^{|v|}(z) \in (X^n)_{(\delta,0)}$, since $\sigma^{|v|}(z)_i = y_i$ for all $i \geq 1$. Consequently, $l^{|v|}(\beta) = \gamma$. Thus l acts transitively, and $\#L$ divides n since $l^n = \mathrm{id}$. When $x \in \mathrm{Per}\, X_{(\alpha,0)}$ there is a $\gamma \in L$ such that $x \in (X^n)_{(\gamma,0)}$. The identity $\sigma^{\mathrm{period}(x)}(x) = x$ implies first that $l^{\mathrm{period}(x)}(\gamma) = \gamma$, and then that $\#L$ divides period(x). □

Lemma 3.9. *In the setting of Lemma 3.8, assume that* $n = \operatorname{period}(X_{(\alpha,0)})$. *Then* $\#L = \operatorname{period}(X_{(\alpha,0)})$.

Proof. Let $A_{\alpha,1} \subseteq A_{\alpha,2} \subseteq \dots$ be the irreducible SFTs of Theorem 3.2. We may assume that $\operatorname{period}(A_{\alpha,l}) = \operatorname{period}(X_{\alpha,0})$ for all $l \in \mathbb{N}$. (3.7) gives us a partition

$$A_{\alpha,l} = \bigsqcup_{\beta \in L} (X^n)_{(\beta,0)} \cap A_{\alpha,l} \tag{3.8}$$

for each $l \in \mathbb{N}$. By construction there is a $K \in \mathbb{N}$ such that

$$(X^n)_{(\beta,0)} \cap A_{\alpha,l} = \{x \in A_{\alpha,l} : \ [x_{[ni,ni+Kn[}] = \beta \text{ in } \mathcal{S}(X^n) \ \ \forall i \in \mathbb{Z}\}.$$

This shows that $(X^n)_{(\beta,0)} \cap A_{\alpha,l}$ is closed for all $\beta \in L$. Note that $(X^n)_{(\beta,0)} \cap A_{\alpha,l}$ is σ^n-invariant. On the other hand, since $A_{\alpha,l}$ is an SFT of period n, there is also a partition

$$(A_{\alpha,l})^n = \bigsqcup_{i=0}^{n-1} D_{i,l},$$

where each $D_{i,l}$ is a mixing SFT, and $\sigma(D_{i,l}) = D_{i+1,l}$ modulo n; cf., e.g., [LM]. It follows that there is a partition $\{0,1,2,\dots,n-1\} = \bigsqcup_{\beta \in L} T_\beta$ such that $(X^n)_{(\beta,0)} \cap A_{\alpha,l} = \bigsqcup_{j \in T_\beta} D_{j,l}$ for all $\beta \in L$. After a renumbering we may assume that $D_{i,l} \subseteq D_{i,l+1}$ for all l and all $i = 0,1,2,\dots,n-1$. Then T_β is l-independent, i.e. $(X^n)_{(\beta,0)} \cap A_{\alpha,l} = \bigsqcup_{j \in T_\beta} D_{j,l}$ for all $l \in \mathbb{N}$. Now assume that $\#T_\beta \geq 2$ for some $\beta \in L$. Pick an $l \in \mathbb{N}$. There are then elements $j_1, j_2 \in T_\beta$ such that $j_1 \neq j_2$, and periodic points $p \in D_{j_1,l}, q \in D_{j_2,l}$. As in the proof of Theorem 3.2, we construct an element $r^1 \in (X^n)_{(\beta,0)}$ such that r^1 is forward asymptotic to the orbit of q and backward asymptotic to the orbit of p (everything under the action of σ^n). Since $(X^n)_{(\beta,0)} = \bigcup_{k \in \mathbb{N}} (X^n)_{(\beta,0)} \cap A_{\alpha,k}$, there is a $k > l$ such that $r^1 \in (X^n)_{(\beta,0)} \cap A_{\alpha,k}$, and we conclude then that p and q are contained in the same $D_{i,k}$. Since $D_{j_1,l} \subseteq D_{j_1,k}$ and $D_{j_2,l} \subseteq D_{j_2,k}$, this is impossible. This contradiction shows that $\#T_\beta = 1$ for all β, proving that $\#L = n$. □

Lemma 3.10. *Let* $X_{(\alpha,0)}$ *be an irreducible component at level* 0 *in the shift space* X. *Let* $\left(\overline{X_{(\alpha,0)}}\right)_c$ *be the top component of the synchronized system* $\overline{X_{(\alpha,0)}}$. *Then*

$$X_{(\alpha,0)} \subseteq \left(\overline{X_{(\alpha,0)}}\right)_c.$$

Proof. The desired inclusion will follow from the definitions if we can show that every word $u \in \alpha$ is synchronizing for $\overline{X_{(\alpha,0)}}$. Therefore let $x, y \in \mathbb{W}\left(\overline{X_{(\alpha,0)}}\right)$ and assume that $xu, uy \in \overline{X_{(\alpha,0)}}$. Since $\overline{X_{(\alpha,0)}} \subseteq R(X)$ and u is synchronizing for $R(X)$, we conclude that $xuy \in \mathbb{W}(R(X))$. There is then a cycle $p \in \mathbb{W}(R(X))$ such that $xuy \subseteq p$. Since $u \in \alpha$, we conclude that $p^\infty \in X_{(\alpha,0)}$. It follows that $xuy \in \mathbb{W}\left(X_{(\alpha,0)}\right) = \mathbb{W}\left(\overline{X_{(\alpha,0)}}\right)$, and we conclude that u is synchronizing for $\overline{X_{(\alpha,0)}}$, as desired. □

We remark that in general the inclusion in Lemma 3.10 is a strict inclusion.

Proposition 3.11. *Let* $X_{(\alpha,0)}$ *be an irreducible component at level* 0 *in the shift space* X. *Set* $p = \operatorname{period}(X_{(\alpha,0)})$. *Then there are closed subsets* D_i, $i = 0,1,2,\dots,$ $p-1$, *in* X *such that*

i) $\overline{X_{(\alpha,0)}} = \bigcup_{i=0}^{p-1} D_i$,
ii) $\sigma(D_i) = D_{i+1}$ *modulo* p,

iii) $\sigma^p|_{D_i}$ *is mixing for all* $i = 0, 1, 2, \ldots, p-1$, *and*
iv) $D_i \cap D_j$ *has empty interior when* $i, j \in \{0, 1, \ldots, p-1\}$ *and* $i \neq j$.

Proof. By Lemma 3.8 and Lemma 3.9 there is a partition

$$X_{(\alpha,0)} = \bigsqcup_{i=0}^{p-1} (X^p)_{(\beta_i,0)}, \tag{3.9}$$

where each $(X^p)_{(\beta_i,0)}$ is an irreducible component at level 0 of X^p, such that

$$\sigma\left((X^p)_{(\beta_i,0)}\right) = (X^p)_{(\beta_{i+1},0)}, \tag{3.10}$$

modulo p. Set $D_i = \overline{(X^p)_{(\beta_i,0)}}$. Then i) and ii) clearly hold. To show that σ^p is mixing on each D_i, observe that it follows from (3.9) and (3.10) that period$((X^p)_{(\beta_i,0)})$ $= 1$. By Lemma 3.10 this implies that the period of the top component of $\overline{(X^p)_{(\beta_i,0)}}$ $= D_i$ is also 1. This in turn implies that $\sigma^p|_{D_i}$ is mixing, by Lemma 3.6.

Should $D_i \cap D_j$ have non-empty interior, it follows that $\bigcup_{k\in\mathbb{N}} \sigma^{kp}(D_i \cap D_j)$ is dense in D_i since σ^p is topologically mixing on D_i. But $\sigma^p(D_i \cap D_j) = D_i \cap D_j$, so we conclude that $D_i = D_i \cap D_j$. By Lemma 3.4 this is only possible when $i = j$ mod p. □

Corollary 3.12. *Let* X *be a synchronized system with top component* X_c. *Set* $p = \mathrm{period}(X_c)$. *There are closed subsets* $D_i \subseteq X$, $i = 0, 1, 2, \ldots, p-1$, *such that*

i) $X = \bigcup_{i=0}^{p-1} D_i$,
ii) $\sigma(D_i) = D_{i+1}$ *modulo* p,
iii) $\sigma^p|_{D_i}$ *is mixing for all* $i = 0, 1, 2, \ldots, p-1$, *and*
iv) $D_i \cap D_j$ *has empty interior when* $i, j \in \{0, 1, \ldots, p-1\}$ *and* $i \neq j$.

Proof. This is a special case of Propostion 3.11. □

The subsets $D_i, i = 0, 1, \ldots, p-1$, of Corollary 3.12 will be called *the cyclic cover of* X. The cyclic cover is unique in the following sense.

Lemma 3.13. *Let* $C_0, C_1, \ldots, C_{m-1}$ *be closed subsets of the synchronized system* X *such that*

a) $X = \bigcup_{i=0}^{m-1} C_i$,
b) $\sigma(C_i) = C_{i+1}$ *modulo* m,
c) $\sigma^m|_{C_i}$ *is mixing for all* i, *and*
d) $i \neq j \Rightarrow C_i \neq C_j$.

Then $m = p$, *where* p *is the period of the top component in* X, *and there is a natural number* j *such that* $C_i = D_{i+j}$ *modulo* p *for all* i, *where* $D_0, D_1, \ldots, D_{p-1}$ *is the cyclic cover of* X.

Proof. Let $i \in \{0, 1, 2, \ldots, p-1\}$. Since σ^{pm} is mixing on D_i, there is a point $z \in D_i$ whose orbit under σ^{pm} is dense in D_i. It follows from a) that $z \in C_j$ for some j, and then from b) that $D_i \subseteq C_j$. By a similar argument, using c), we find that $C_j \subseteq D_{i'}$ for some i'. It follows from iv) of Lemma 3.12 that $D_i = D_{i'}$, and hence that $D_i = C_j$. It follows from d) that there is a bijection $\mu : \{0, 1, \ldots, p-1\} \to \{0, 1, \ldots, m-1\}$ such that $D_i = C_{\mu(i)}$ for all $i \in \{0, 1, \ldots, m-1\}$. In particular, $m = p$, and if we choose $j \in \{0, 1, \ldots, m-1\}$ such that $D_j = C_0$, then ii) of Lemma 3.12 and condition b) ensure that $C_i = D_{i+j}$ modulo p for all i. □

4. Deeper-lying irreducible components

We set

$$\partial X = \{x \in R(X) : \ w \subseteq x \Rightarrow w \notin \mathbb{S}(X)\},$$

and call it the *derived shift space* of X. Since ∂X is a shift space we can continue, and consider $\partial(\partial X) = \partial^2 X, \ \partial(\partial^2 X) = \partial^3 X$, etc. Of course, it can happen that these constructions give nothing interesting; it may be that there are no periodic points in X, in which case $\partial X = \emptyset$ since $R(X) = \emptyset$, or it can be that there are no synchronizing words for $R(X)$, in which case $\partial X = R(X)$. When X is an SFT, $\partial X = \emptyset$ because all sufficiently long words occurring in $R(X)$ are synchronizing for $R(X)$. For convenience we set $\partial^0 X = X$. We call $\partial^k X$ *the kth derived shift space* of X. We define the *depth* of X to be

$$\mathrm{Depth}(X) = \sup\{k \in \mathbb{N} : \partial^k X \neq \emptyset\}.$$

Thus a minimal shift space with infinitely many points as well as an SFT have depth 0, but for different reasons. We will show in Section 6 that sofic shift spaces have finite depth, and that the depth of an irreducible sofic shift space can be any natural number. In Example 4.8 we describe a synchronized system of infinite depth. Example 0.10 of [FF] gives examples of coded shift spaces which are not synchronized, i.e., do not have any synchronizing words. Such shift spaces also have infinite depth.

Example 4.1. The context free shift space is the subshift Z of $\{a, b, c\}^{\mathbb{Z}}$ obtained by disallowing the words $ab^i c^j a$ when $i \neq j$. It is easy to see that $w \in \mathbb{W}(Z)$ is synchronizing for Z if and only if $a \subseteq w$ or $cb \subseteq w$. For any pair of words $w, u \in \mathbb{W}(Z)$ we can find (possibly empty) words $s, t \in \mathbb{W}(Z)$ such that $wsatu \in \mathbb{W}(Z)$, proving that Z is irreducible. Similarly, for any $w \in \mathbb{W}(Z)$ there are words $s, t \in \mathbb{W}(Z)$ such that $aswta \in \mathbb{W}(Z)$. As a is synchronizing, $w \subseteq (aswt)^\infty$, proving that $R(Z) = \overline{\mathrm{Per}\, Z} = Z$. It follows that Z contains a single irreducible component at level 0 and that ∂Z is an SFT whose non-wandering part consists of the fixed points b^∞ and c^∞. It follows that $\mathrm{Depth}(Z) = 1$.

When X is a shift space of finite depth k, we call $\partial^k X$ the *floor* of X. Note that if $\partial^k X$ is the floor of X, then $R(\partial^k X)$ is a (possibly empty) SFT.

The irreducible components at level 0 of $\partial^k X$ will be called *the irreducible components of X at level k.* They will be denoted by $X_{(\alpha,k)}, \alpha \in \mathcal{S}(\partial^k X)$.

Clearly, every irreducible component at level 0 of a shift space X is contained in $R(X)\backslash\partial X$. In general the irreducible components do not exhaust $R(X)\backslash\partial X$, not even when X is irreducible and sofic; unless X is an SFT, a transitive point is not contained in an irreducible component. However, the irreducible components do contain all periodic points of $X\backslash\partial X$:

Lemma 4.2. *Every periodic point in $X\backslash\partial X$ is contained in a unique irreducible component at level 0. If X has finite depth, every periodic point in X is contained in a unique irreducible component at some level.*

Proof. Let $p \in \mathbb{W}(X)$ be a cycle such that $p^\infty \notin \partial X$. There is then an $N \in \mathbb{N}$ so large that p^N contains a word w which is synchronizing for $R(X)$. Hence $p^\infty \in X_{([w],0)}$. By Lemma 3.4, $X_{([w],0)}$ is the only irreducible component at level 0 that contains p^∞.

If X is of finite depth and $x \in \operatorname{Per} X$, set $k = \min\{j : x \in \partial^j X\}$. It follows from the first part of the lemma that x is contained in a unique irreducible component at level k. Since the levels $\partial^j X \backslash \partial^{j+1} X, j = 0, 1, \dots, \operatorname{Depth}(X)$, are mutually disjoint, this irreducible component is also unique among all irreducible components. □

Lemma 4.3. *Let $X_{(\alpha,0)}, X_{(\beta,0)}$ be irreducible components at level* 0 *in X. Assume that $A \subseteq X$ is an irreducible SFT such that $A \cap X_{(\alpha,0)} \neq \emptyset$ and $A \cap X_{(\beta,0)} \neq \emptyset$. Then $\alpha = \beta$.*

Proof. Let $x \in X_{(\alpha,0)} \cap A, x' \in X_{(\beta,0)} \cap A$. There is then an element $z \in A$ which is backward asymptotic to x and forward asymptotic to x'. By definition of $X_{(\alpha,0)}$ and $X_{(\beta,0)}$ there is an $N \in \mathbb{N}$ such that $x_{[i,i+N[}$ contains an element from α for all $i \in \mathbb{Z}$, and such that $x'_{[i,i+N[}$ contains an element from β for all $i \in \mathbb{Z}$. It follows that z contains words from both α and β, and this implies that $\alpha = \beta$ since $z \in A \subseteq R(X)$. □

The next aim is to show that the derived shift spaces and their structure of irreducible components is a conjugacy invariant for a shift space.

Lemma 4.4. *Let X and Y be shift spaces. Let $\varphi : X \to Y$ be a conjugacy. Then there is a natural number N so large that*

$$v \in \mathbb{W}(R(X)), \varphi(v) \in \mathbb{S}(Y), |v| \geq N \Rightarrow v \in \mathbb{S}(X).$$

Proof. The proof is the same as the proof of Theorem 2.1.10 in [LM]: By choosing $l \in \mathbb{N}$ large enough we can assume that both φ and φ^{-1} have memory and anticipation l. Then $\varphi^{-1} \circ \varphi : \mathbb{W}_{4l+1}(R(X)) \to \mathbb{W}_1(R(X))$ is the map which picks out the central letter in the word. Let $v \in \mathbb{W}(R(X))$ be such that $|v| \geq 4l$ and $\varphi(v) \in \mathbb{S}(Y)$. We claim that $v \in \mathbb{S}(X)$. To see this, let $u, w \in \mathbb{W}(R(X))$ such that $uv, vw \in \mathbb{W}(R(X))$. Choose $s, t \in \mathbb{W}_{2l}(R(X))$ such that $suv, vwt \in \mathbb{W}(R(X))$. Although we don't know that $suvwt \in \mathbb{W}(X)$, we can make sense of $\varphi^{-1} \circ \varphi(suvwt)$ since every $(4l+1)$-block in $suvwt$ is in $\mathbb{W}_{4l+1}(R(X))$, and we find that $\varphi^{-1} \circ \varphi(suvwt) = uvw$. Now $\varphi(suv) = u'\varphi(v)$ and $\varphi(vwt) = \varphi(v)w'$ for some $u', w' \in \mathbb{W}(R(Y))$. Since $\varphi(suv), \varphi(vwt) \in \mathbb{W}(R(Y))$, we conclude that $u'\varphi(v)w' \in \mathbb{W}(R(Y))$ since $\varphi(v) \in \mathbb{S}(Y)$. But then $uvw = \varphi^{-1}(\varphi(suvwt)) = \varphi^{-1}(u'\varphi(v)w') \in \mathbb{W}(R(X))$, proving the claim. Hence $N = 4l$ works. □

Proposition 4.5. *Let $\varphi : X \to Y$ be a conjugacy of shift spaces. Then $\varphi(\partial^k X) = \partial^k Y$ for all $k = 0, 1, 2, 3, \dots$. In particular,* $\operatorname{Depth} X = \operatorname{Depth} Y$.

Proof. Note first that $\varphi(R(X)) = R(Y)$. It follows from Lemma 4.4 that $x \in R(X)$ contains a word which is synchronizing for $R(X)$ if $\varphi(x)$ contains a word which is synchronizing for $R(Y)$. This means that $\varphi^{-1}(R(Y) \backslash \partial Y) \subseteq R(X) \backslash \partial X$. Thus $\partial X \subseteq \varphi^{-1}(\partial Y)$. By symmetry $\partial Y \subseteq \varphi(\partial X)$, and we conclude that $\varphi(\partial X) = \partial Y$. Proceed by induction. □

Proposition 4.6. *Let X and Y be shift spaces and $\varphi : X \to Y$ a conjugacy. For every irreducible component $X_{(\alpha,k)}$ at level k in X, there is an irreducible component $Y_{(\beta,k)}$ at level k in Y such that $\varphi\left(X_{(\alpha,k)}\right) = Y_{(\beta,k)}$.*

Proof. By Proposition 4.5 we may assume that $k = 0$. Since $\varphi(R(X)) = R(Y)$, we can also assume that $X = R(X)$ and $Y = R(Y)$. Consider an irreducible component

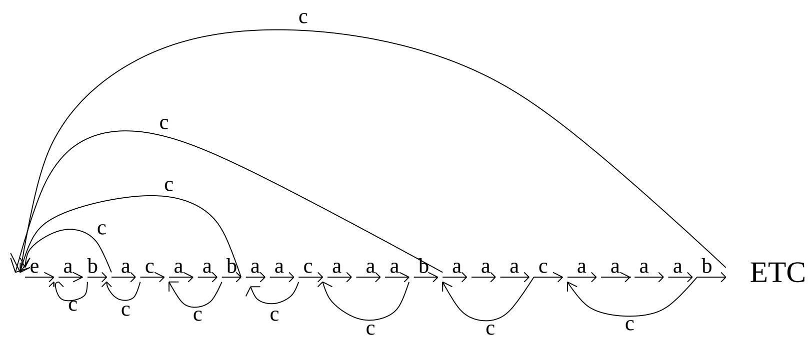


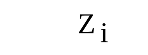
FIGURE 2.

$Y_{(\beta,0)}$ at level 0 in Y. Assume that φ has anticipation and memory both equal to $l \in \mathbb{N}$. Let $y \in Y_{(\beta,0)}$. There is then an $M \in \mathbb{N}$ such that $y_{[i,i+n[} \in \beta$ for all $i \in \mathbb{Z}$ and all $n \geq M$. It follows from Lemma 4.4 that there is an $N \in \mathbb{N}$, $N \geq M$, such that $\varphi^{-1}(y)_{[i-l,i+n+l+1[} \in \mathbb{S}(X)$ for all $i \in \mathbb{Z}$, when $n \geq N$. Consider an $n \geq N$. Since there are only finitely many elements in $\mathbb{W}_{n+2l+1}(X)$, there must be two elements $w_+, w_- \in \mathbb{S}(X) \cap \mathbb{W}_{n+2l+1}(X)$ and integers $i_- \leq i_+$ such that $\left[\varphi^{-1}(y)_{[i-l,i+n+l+1[}\right] = [w_-] \in \mathcal{S}(X)$ for all $i \leq i_-$ and $\left[\varphi^{-1}(y)_{[i-l,i+n+l+1[}\right] = [w_+] \in \mathcal{S}(X)$ for all $i \geq i_+$. Since $\varphi^{-1}(y)$ can be approximated arbitrarily well with periodic points, there is a periodic element $q \in X$ such that $\varphi^{-1}(y)_{[i_- -l,i_+ +n+l+1[} = q_{[i_- -l,i_+ +n+l+1[}$. Thus $q \in X$ is a periodic point containing a block of the form $x_- y x_+$, where $y \in \mathbb{W}(X)$, $x_-, x_+ \in \mathbb{S}(X)$ and $[x_-] = [w_-]$ and $[x_+] = [w_+]$ in $\mathcal{S}(X)$. It follows that $[w_+] = [w_-]$ in $\mathcal{S}(X)$. Thus, for a given $y \in Y_{(\beta,0)}$ there is an irreducible component $X_{(\alpha_y,0)}$ in X such that $\varphi^{-1}(y) \in X_{(\alpha_y,0)}$. When $y, y' \in Y_{(\beta,0)}$, there is an irreducible SFT $A \subseteq Y_{(\beta,0)}$ containing both y and y' by Theorem 3.2. Then $\varphi^{-1}(A)$ is an irreducible SFT in X containing both $\varphi^{-1}(y)$ and $\varphi^{-1}(y')$. It follows from Lemma 4.3 that $\alpha_y = \alpha_{y'}$. There is therefore an element $\alpha \in \mathcal{S}(X)$ such that $\varphi^{-1}(Y_{(\beta,0)}) \subseteq X_{(\alpha,0)}$. By symmetry there is an element $\gamma \in \mathcal{S}(Y)$ such that $\varphi(X_{(\alpha,0)}) \subseteq Y_{(\gamma,0)}$. Since $Y_{(\beta,0)} = \varphi(\varphi^{-1}(Y_{(\beta,0)})) \subseteq Y_{(\delta,0)}$, it follows from Lemma 3.4 that $\beta = \delta$. Hence $\varphi\left(X_{(\alpha,0)}\right) = Y_{(\beta,0)}$. □

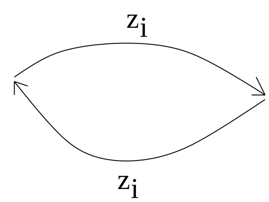
Example 4.7. We give an example of a synchronized system of depth 2, which contains infinitely many irreducible components at level 1. Consider the synchronized system presented by the infinite labeled graph G shown in Figure 2.

The shift space, whose finite words in the alphabet $\{a, b, c, e\}$ are those which label a finite path in G, is a synchronized system. The derived shift space ∂X consists of the periodic points $(a^k c)^\infty, k = 1, 2, \ldots$, and the fixed point a^∞. Except for the fixed point, each of the periodic orbits forms an irreducible component at level 1 in X. Since $\partial^2 X = \{a^\infty\}$, we see that $\operatorname{Depth} X = 2$.

Example 4.8. There are also synchronized systems of infinite depth such that all the derived shift spaces ∂X, $\partial^2 X$, $\partial^3 X, \ldots$ are synchronized systems. To describe a synchronized system in $\{a, b, c\}^{\mathbb{Z}}$ with this property, let $x_1, x_2, \ldots$ be a numbering of the words in $\{a, b\}^{\mathbb{Z}}$ and set $z_i = c x_i c \in \mathbb{W}\left(\{a, b, c\}^{\mathbb{Z}}\right), i = 1, 2, \ldots$. To describe the system we have in mind, let



be short-hand for the oriented labeled graph



With this convention Figure 3 is an infinite oriented graph labeled by the z_i's. The words in the alphabet $\{a, b, c\}$ which label the finite walks in this graph are the words of a shift space X, which is a synchronized system since the graph is irreducible and the words $z_1z_2, z_2z_1, z_2z_i, z_iz_2, z_1z_i, z_iz_1, i = 3, 4$, are synchronizing for X. It is easy to see that an element of X contains a synchronizing word if and only if it contains one of these words, i.e. if and only if it 'visits the top'. It follows therefore that the derived shift space ∂X of X is given, with the same convention, by the labeled graph of Figure 4.

This can be repeated indefinitely, so we see that $\partial^n X$ is a non-empty synchronizing shift space for all $n \in \mathbb{N}$.

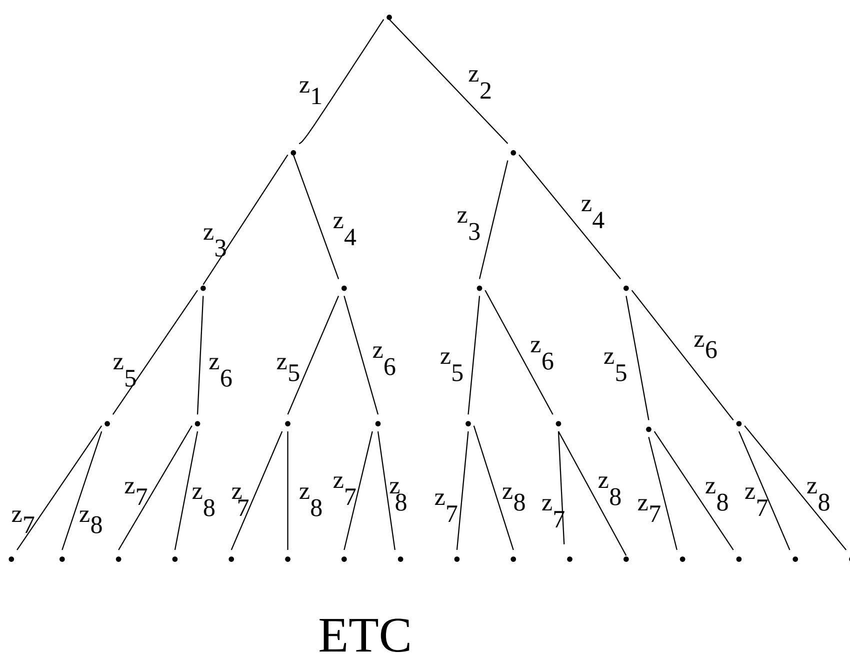


Figure 3.

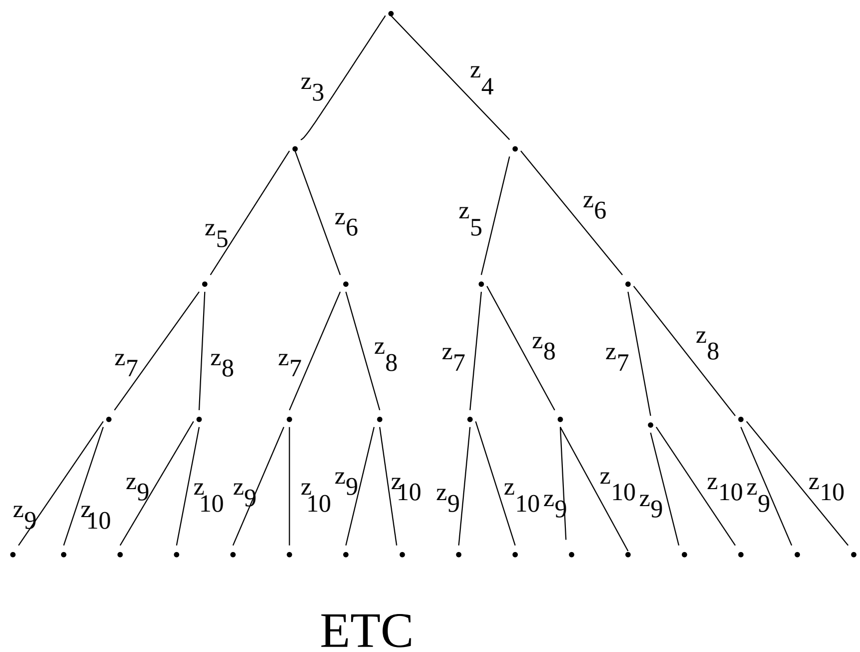


FIGURE 4.

5. AFFILIATION

Let X be a shift space. Let $X_{(\alpha,k)}$ be an irreducible component of X at level k. We say that a periodic point $x \in X$ is *d-affiliated* to $X_{(\alpha,k)}$ when $x \in \partial^k X$ and there are words $w, u \in \alpha$ such that $w\left(x_{[0,\text{period}(x)[}\right)^{di} u \in \mathbb{S}(\partial^k X)$ for all $i = 1, 2, 3, \ldots$. The set of periodic points that are d-affiliated to $X_{(\alpha,k)}$ will be denoted by $X^{(d)}_{(\alpha,k)}$.

Lemma 5.1. *If $x \in \operatorname{Per} X$ is d-affiliated to $X_{(\alpha,k)}$, then $\sigma^j(x)$ is d-affiliated to $X_{(\alpha,k)}$ for all $j \in \mathbb{Z}$.*

Proof. Set $p = \text{period}(x)$. We may assume that $0 \leq j < p$. There are words $w, u \in \alpha$ such that $w\left(x_{[0,p[}\right)^{di} u \in \mathbb{S}(\partial^k X)$ for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$ and set $w' = wx_{[0,j[}, u' = x_{[j,dp[}u$. Note that $u', w' \in \mathbb{S}(\partial^k X)$. Since

$$w'\left(\sigma^j(x)_{[0,p[}\right)^{di} u' = w\left(x_{[0,p[}\right)^{d(i+1)} u \in \mathbb{S}(\partial^k X),$$

it suffices to show that $u', w' \in \alpha$. Choose $b \in \mathbb{W}(R\left(\partial^k X\right))$ such that $ubw \in \mathbb{S}(\partial^k X)$. By using that u and w are synchronizing for $R\left(\partial^k X\right)$ we find that $\left(w\left(x_{[0,p[}\right)^d ub\right)^\infty \in \operatorname{Per} \partial^k X$. Since u', w' and u all occur in $\left(w\left(x_{[0,p[}\right)^d ub\right)^\infty$, we conclude that $w' \sim u' \sim u$ in $\mathbb{S}\left(\partial^k X\right)$. ☐

Let p be a minimal cycle in $\mathbb{W}(X)$, and $X_{(\alpha,k)}$ an irreducible component in X at level k. Thanks to Lemma 5.1 it makes sense to say that p is *d-affiliated* to $X_{(\alpha,k)}$ when p^∞ is d-affiliated to $X_{(\alpha,k)}$.

Lemma 5.2. *Let p be a minimal cycle in $\mathbb{W}(X)$, and $X_{(\alpha,k)}$ an irreducible component in X at level k. Assume that p is 1-affiliated to $X_{(\alpha,k)}$. Let $A \subseteq \partial^k X$ be an* *irreducible SFT such that $\mathbb{W}(A) \cap \alpha \neq \emptyset$. Then there is an irreducible SFT B such that $A \cup \{p^\infty\} \subseteq B \subseteq \partial^k X$.*

Proof. Let s, t be elements of α such that $sp^it \in \partial^k X$ for all $i \in \mathbb{N}$. Since $X_{(\alpha,k)}$ is irreducible by Theorem 3.2, there is a word $r \in \mathbb{W}(X_{(\alpha,k)})$ such that $trs \in \mathbb{W}(X_{(\alpha,k)})$. Then $w = trs \in \mathbb{W}(X_{(\alpha,k)})$ is a synchronizing word for $\partial^k X$ such that $wp^iw \in \mathbb{W}(\partial^k X)$ for all $i \in \mathbb{N}$. Let $w' \in \alpha \cap \mathbb{W}(A)$. Since $X_{(\alpha,k)}$ is irreducible by Theorem 3.2, there are words $x, y \in \mathbb{W}(X_{(\alpha,k)})$ such that $w'xw, wyw' \in \mathbb{W}(X_{(\alpha,k)})$. It follows that

$$wp^{i_1}wyw'a_1w'xwp^{i_2}wyw'a_2w'xwp^{i_3}\dots a_nw'xwp^{i_{n+1}} \in \mathbb{W}(\partial^k X) \tag{5.1}$$

for all $n \in \mathbb{N}$, all $i_1, i_2, \dots, i_{n+1} \in \mathbb{N}$ and all $a_1, a_2, \dots, a_n \in \mathbb{W}(A)$, provided $w'a_iw' \in \mathbb{W}(A), i = 1, 2, \dots, n$. Set $u = wyw', v = w'xw$. We can assume that $p^\infty \notin A$. Then there is an N so large that

$$z \nsubseteq vp^\infty \tag{5.2}$$

and

$$z \nsubseteq p^\infty u \tag{5.3}$$

when $z \in \mathbb{W}(A)$ and $|z| \geq N$, while

$$z \nsubseteq a_1 v \tag{5.4}$$

and

$$z \nsubseteq ua_1 \tag{5.5}$$

when $a_1 \in \mathbb{W}(A), z \subseteq p^\infty$ and $|z| \geq N$. Let n be a step-length of A. We may assume that $N > \max\{n, |p|, |w'|\}$. Set $L = 6N + |v| + |p| + |u|$. We construct an SFT B_0 as follows. The states of B_0 consist of the words of length L which occur in a right-infinite stretch in $\mathcal{A}^{\mathbb{Z}}$ of the form

$$avp^\infty \qquad \text{(called type 1)},$$

where $aw' \in \mathbb{W}(A)$, or a left-infinite stretch of the form

$$p^\infty ua \qquad \text{(called type 2)},$$

where $w'a \in \mathbb{W}(A)$. A state $x_1x_2\dots x_L$ in B_0 can follow another state $y_1y_2\dots y_L$ when $y_2y_3\dots y_L = x_1x_2\dots x_{L-1}$. Let $\mathcal{A}$ be the states of X and define an embedding $\varphi : B_0 \to \mathcal{A}^{\mathbb{Z}}$ by

$$\varphi((a_ia_i^1\dots a_i^{L-1})_{i\in\mathbb{Z}}) = (a_i)_{i\in\mathbb{Z}}.$$

We claim that $\varphi(B_0) \subseteq \partial^k X$. To prove this, let $x \in \varphi(B_0)$ and $i_0 \in \mathbb{Z}$. It suffices to show that $x_{[i_0,n[} \in \mathbb{W}(\partial^k X)$ for all $n > i_0$. Any state of B_0 either begins or ends with a word of length at least $3N$ which either is in $\mathbb{W}(A)$, or is a subword of p^∞. We can therefore choose $j \leq i_0$ such that $x_{[j,j+3N[} \subseteq p^\infty$ or $x_{[j,j+3N[} \in \mathbb{W}(A)$. Assume first that $x_{[j,j+3N[} \in \mathbb{W}(A)$. Set $j_1 = \sup\{n > j : x_{[j,n[} \in \mathbb{W}(A)\}$. If $j_1 = \infty$, we are done, so assume that $j_1 < \infty$. By definition of $\varphi(B_0)$ there is a state s of B_0 such that $s = x_{[j_1-3N,j_1-3N+L[}$. Note that s must be of type 1, because of (5.3) and because s 'leaves' $\mathbb{W}(A)$—recall that N is larger than a step-length of A. It follows that $s = a_1vpl_p$ for some word $a_1w' \in \mathbb{W}(A)$ and some word $pl_p \subseteq p^\infty$ with $|l_p| \geq 3N$ and $|a_1| \geq 2N > n$. Hence $x_{[j,j_1-3N+L[}$ has the form a_0vpl_p with $a_0w' \in \mathbb{W}(A)$, $l_p \subseteq p^\infty$ with $|l_p| \geq 3N$. Set $j_2 = \sup\{n \geq j_1 - 3N + 1 + L : x_{[j,n[} = a_0vpl$ for some $l \subseteq p^\infty\}$. If $j_2 = \infty$, we see from (5.1) that $x_{[j,j'[} \in \mathbb{W}(\partial^k X)$ for all $j' \geq j$, and we are done. If $j_2 < \infty$, we continue as follows: By definition of $\varphi(B_0)$, there is a state s of B_0 such that $s = x_{[j_2-3N, j_2-3N+L[}$. Note that s must be of type 2 because of (5.4) and because s 'leaves' p^∞—we use here that $N \geq |p|$. Consequently, $s = l'pua$, where $l' \subseteq p^\infty$, $w'a \in \mathbb{W}(A)$ and $|a| \geq 3N$. It follows that $x_{[j,j_2-3N+L[} = a_0 v p l'_p p u a$, where $p l'_p p \subseteq p^\infty$ and $w'a \in \mathbb{W}(A)$ with $|a| \geq 3N$. Note that $p l'_p p$ must have the form p^{i_1} for some $i_1 \in \mathbb{N}$, so that $x_{[j,j_2-3N+L[}$ has the form $a_0 v p^{i_1} u a$, where $a_0 w', w'a \in \mathbb{W}(A)$ with $|a| \geq 3N$, and $i_1 \in \mathbb{N}$. In particular, we see from (5.1) that $x_{[j,j_2-3N+L[} \in \mathbb{W}(\partial^k X)$. We can now proceed by induction to obtain an increasing sequence $n_1 < n_2 < \ldots$ in $\mathbb{N}$ such that $x_{[j,n_i[} \in \mathbb{W}(\partial^k X)$ for all i. This yields the desired conclusion. A similar argument works in the case where $x_{[j,j+3N[} \subseteq p^\infty$, so we conclude that $\varphi(B_0) \subseteq \partial^k X$. Note that B_0 is irreducible because A is, and that $A \cup \{p^\infty\} \subseteq \varphi(B_0)$. We can therefore use $B = \varphi(B_0)$. □

Lemma 5.3. *Let p be a minimal cycle in $\mathbb{W}(X)$, and $X_{(\alpha,k)}$ an irreducible component in X at level k. Then p is 1-affiliated to $X_{(\alpha,k)}$ if and only if there is an irreducible SFT $B \subseteq \partial^k X$ such that $\mathbb{W}(B) \cap \alpha \neq \emptyset$ and $p^\infty \in B$.*

Proof. Assume first that such an SFT B exists. Let $M \in \mathbb{N}$ be a step-length for B. Let $w \in \mathbb{W}(B) \cap \alpha$. Since B is irreducible, we can find $x_0, y \in \mathbb{W}(B)$ such that $w x_0 p^M y w \in \mathbb{W}(B)$. Since $p^{M+i} \in \mathbb{W}(B)$, we see that $w x_0 p^{M+i} y w \in \mathbb{W}(B)$ for all $i \in \mathbb{N}$. Thus $w x p^i y w \in \mathbb{W}(B)$ for all $i \in \mathbb{N}$ when we set $x = x_0 p^M$. Furthermore, $wx, yw \in \mathbb{W}(B) \subseteq \mathbb{W}(R\left(\partial^k X\right))$, and both words are synchronizing for $R\left(\partial^k X\right)$ since w is. Since B is irreducible, we automatically have $[wx] = [yw] = [w] = \alpha$ in $\mathcal{S}(\partial^k X)$.

For the converse, let A be any non-empty irreducible SFT contained in $X_{(\alpha,k)}$, cf. Theorem 3.2, and use Lemma 5.2. □

In particular, it follows that $\operatorname{Per} X_{(\alpha,k)} \subseteq X^{(1)}_{(\alpha,k)}$.

Lemma 5.4. $\bigcup_{d\in\mathbb{N}} X^{(d)}_{(\alpha,k)} \subseteq \overline{X_{(\alpha,k)}}$.

Proof. When p is a minimal cycle in $\partial^k X$ and $w, u \in \alpha$ are such that $w p^{di} u \in \mathbb{W}(R(X))$ for all $i \in \mathbb{N}$, choose $x \in \mathbb{W}(R(X))$ such that $uxw \in \mathbb{W}(R(X))$. Then $\left(w p^{di} u x\right)^\infty \in X_{(\alpha,k)}$ for all $i \in \mathbb{N}$, proving that $p^\infty \in \overline{X_{(\alpha,k)}}$. □

Lemma 5.5. *Let $X_{(\alpha,k)}$ be an irreducible component at level k in the shift space X. Let $p \in \mathbb{W}(X)$ be a minimal cycle such that p^∞ is d-affiliated to $X_{(\alpha,k)}$. Let $X_{(\alpha,k)} = \bigsqcup_{\beta\in L} \left(X^n\right)_{(\beta,k)}$ be the decomposition (3.7) corresponding to $n = |p|d$. Then there is an $\gamma \in L$ such that $p^d \in \mathbb{W}\left(X^{|p|d}\right)$ is 1-affiliated to $\left(X^{|p|d}\right)_{(\gamma,k)}$.*

Proof. Since p^∞ is d-affiliated to $X_{(\alpha,k)}$, there are words $w_1, w_2 \in \alpha$ such that $w_1 p^{di} w_2 \in \mathbb{S}(\partial^k X)$ for all $i \in \mathbb{N}$. Since $[w_1] = [w_2]$ in $\mathcal{S}\left(\partial^k X\right)$, there are words $a, b \in \mathbb{W}(\partial^k X)$ such that $(w_1 a w_2 b)^\infty \in \partial^k X$. It follows that $(w_1 a w_2 b)^\infty$ contains words $w'_i \in \mathbb{S}\left(\partial^k X^{|p|d}\right), i = 1, 2$, such that $[w'_i] = [w_i] = \alpha$ in $\mathcal{S}(\partial^k X), i = 1, 2$, and

$$w'_1 p^{di} w'_2 \in \mathbb{W}\left(R\left(\partial^k X^{|p|d}\right)\right) \tag{5.6}$$

for all $i \in \mathbb{N}$. (We are here using that $\partial^k\left(X^{|p|d}\right) = \left(\partial^k X\right)^{|p|d}$.) Since $[w'_1] = [w'_2]$ in $\mathcal{S}\left(\partial^k X\right)$, there is a word $y \in \mathbb{W}\left(R\left(\partial^k X\right)\right)$ such that $w'_2 y w'_1 \in \mathbb{S}\left(\partial^k X\right)$. It follows that $(w'_1 p^d w'_2 y)^\infty \in X_{(\alpha,k)}$, because $[w'_1] = [w'_2] = \alpha$. By Lemma 3.8 there is a

$\gamma \in L$ such that $(w_1' p^d w_2' y)^\infty \in \left(X^{|p|d}\right)_{(\gamma,k)}$. Hence $[w_1'] = [w_2'] = \gamma$ by Lemma 3.3 applied to $\partial^k X^{|p|d}$. Then (5.6) tells us that $p^d \in \mathbb{W}\left(X^{|p|d}\right)$ is 1-affiliated to $\left(X^{|p|d}\right)_{(\gamma,k)}$. □

Proposition 5.6. *Let $\varphi : X \to Y$ be a conjugacy of shift spaces, $X_{(\alpha,k)}$ an irreducible component at level k in X, and $Y_{(\beta,k)}$ the irreducible component at level k in Y such that $Y_{(\beta,k)} = \varphi\left(X_{(\alpha,k)}\right)$; cf. Proposition 4.6. Then*

$$\varphi\left(X^{(d)}_{(\alpha,k)}\right) = Y^{(d)}_{(\beta,k)}$$

for all $d \in \mathbb{N}$.

Proof. Let $p \in \mathbb{W}(X)$ be a minimal cycle d-affiliated to $X_{(\alpha,k)}$. By Lemma 5.5 there is an irreducible compoment, $\left(X^{|p|d}\right)_{(\alpha',k)}$, in the decomposition $X_{(\alpha,k)} = \bigsqcup_{\beta \in L} \left(X^{|p|d}\right)_{(\beta,k)}$ of Lemma 3.8 such that p^∞ is 1-affiliated to $\left(X^{|p|d}\right)_{(\alpha',k)}$. By Lemma 5.2 there is an irreducible SFT $A \subseteq \partial^k \left(X^{|p|d}\right)$ containing p^∞ and a periodic point $x \in \left(X^{|p|d}\right)_{(\alpha',k)}$. Then $\varphi(A) \subseteq \partial^k \left(Y^{|p|d}\right)$ by Proposition 4.5, and $\varphi(x) \in \left(Y^{|p|d}\right)_{(\beta',k)}$ for some irreducible component $\left(Y^{|p|d}\right)_{(\beta',k)}$ from the decomposition $Y_{(\beta,k)} = \bigsqcup_{\beta' \in L'} \left(Y^{|p|d}\right)_{(\beta',k)}$ of Lemma 3.8. It follows from Lemma 5.3 that $\varphi\left(p^\infty\right)$ is 1-affiliated to $\left(Y^{|p|d}\right)_{(\beta',k)}$. In other words, when $q = \varphi(p^\infty)_{[0,|p|[}$, there are words $u_1, u_2 \in \beta'$ such that $u_1 q^{di} u_2 \in \mathbb{W}\left(R\left(\partial^k Y\right)\right)$ for all $i \in \mathbb{N}$. It follows that q^∞ is d-affiliated to $Y_{(\beta,k)}$. This shows that $\varphi\left(X^{(d)}_{(\alpha,k)}\right) \subseteq Y^{(d)}_{(\beta,k)}$. By symmetry we must have equality. □

Lemma 5.7. *Let X be a shift space, and $X_{(\alpha,k)}$ an irreducible component of X at level k. Then there is a sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ of irreducible SFTs in $\partial^k X$ such that*

1) $\mathbb{W}(A_n) \cap \alpha \neq \emptyset$ *for all $n \in \mathbb{N}$,*
2) $\operatorname{period}(A_n) = \operatorname{period}\left(X^{(1)}_{(\alpha,k)}\right)$ *for all $n \in \mathbb{N}$,*
3) $\lim_{n\to\infty} h(A_n) = h_{syn}\left(\overline{X_{(\alpha,k)}}\right)$,
4) $X^{(1)}_{(\alpha,k)} = \operatorname{Per}\left(\bigcup_{n=1}^\infty A_n\right)$, *and*
5) $X_{(\alpha,k)} \subseteq \bigcup_{n=1}^\infty A_n \subseteq \overline{X_{(\alpha,k)}}$.

Proof. Let $a_1, a_2, a_3, \dots$ be a numbering of the periodic points 1-affiliated to $X_{(\alpha,k)}$. For each i, let p_i be a minimal cycle such that $p_i^\infty = a_i$. For each i there are $s_i, t_i \in \alpha$ such that $s_i p_i^d t_i \in \mathbb{W}\left(\partial^k X\right)$ for all $d \in \mathbb{N}$. It follows from Theorem 3.2 that there is a sequence $C_1 \subseteq C_2 \subseteq C_3 \subseteq \dots$ of irreducible SFTs in $X_{(\alpha,k)}$ such that

a) $\lim_{n\to\infty} h(C_n) = h_{syn}\left(\overline{X_{(\alpha,0)}}\right)$, and
b) $X_{(\alpha,k)} = \bigcup_n C_n$.

In the following we use the symbol $\langle C, a\rangle$ for the irreducible SFT obtained as in the proof of Lemma 5.2 from an irreducible SFT $C \subseteq \partial^k X$ and a periodic point $a \in X$ which is 1-affiliated to C. Since $C_n \subseteq C_{n+1}$, we can arrange that $\langle C_n, a_1\rangle \subseteq \langle C_{n+1}, a_1\rangle$. This requires only that the words r, w', x and y occurring in the construction of Lemma 5.2 are chosen to be the same in both cases. This is clearly possible; as far as w' is concerned we simply choose it from $\mathbb{W}(C_1)$. By the same reasoning we can arrange that $\langle\langle C_n, a_1\rangle, a_2\rangle \subseteq \langle\langle C_{n+1}, a_1\rangle, a_2\rangle$. By repeating the construction we obtain an increasing sequence $C'_1 \subseteq C'_2 \subseteq \dots$ of irreducible SFTs in $\partial^k X$ such that $C_n \cup \{a_1, a_2, \dots, a_n\} \subseteq C'_n$ for all n. There is an $N \in \mathbb{N}$ such that the greatest common divisor of the periods of $a_1, a_2, \dots, a_N$ is period$(X^{(1)}_{(\alpha,k)})$. Set $A_i = C'_{N+i}$. Then 1) and 2) clearly hold, and $X^{(1)}_{(\alpha,k)} \subseteq \mathrm{Per}\left(\bigcup_{n=1}^{\infty} A_n\right)$. It follows from Lemma 5.3 that $\mathrm{Per}\left(\bigcup_{n=1}^{\infty} A_n\right) \subseteq X^{(1)}_{(\alpha,k)}$, and hence 4) follows. It follows from Lemma 5.3 and Lemma 5.4 that $A_n \subseteq \overline{X_{(\alpha,k)}}$, so that $h(A_n) = h_{syn}(A_n) \leq h_{syn}\left(\overline{X_{(\alpha,k)}}\right)$ for all n. Since $h(A_i) \geq h(C_i)$, 3) follows from a). Since $\bigcup_n C_n \subseteq \bigcup_n A_n$, 5) follows from b). □

Lemma 5.8. *Let X be an irreducible sofic shift space with top component X_c. Set $p = \mathrm{period}(X_c)$. Then*

$$h(X) = \lim_{n\to\infty} \frac{1}{np} \log q_{np}(X) = \lim_{n\to\infty} \frac{1}{np} \log q_{np}\left(X_c^{(1)}\right) = \lim_{n\to\infty} \frac{1}{np} \log q_{np}(X_c)$$
$$= \lim_{n\to\infty} \frac{1}{np} \log P_{np}(X) = \lim_{n\to\infty} \frac{1}{np} \log P_{np}\left(X_c^{(1)}\right) = \lim_{n\to\infty} \frac{1}{np} \log P_{np}(X_c).$$

Proof. The minimum of the numbers $q_{np}(X), q_{np}\left(X_c^{(1)}\right), q_{np}(X_c), P_{np}(X)$, $P_{np}\left(X_c^{(1)}\right), P_{np}(X_c)$ is $q_{np}(X_c)$ and the maximum is $P_{np}(X)$. Since

$$h(X) \geq \limsup_j \frac{1}{j} \log P_j(X)$$

by Proposition 4.1.15 of [LM], it therefore suffices to show that

$$\liminf_n \frac{1}{np} \log q_{np}(X_c) \geq h(X).$$

To this end let $\epsilon > 0$. It follows from Lemma 3.1 and Theorem 3.2 that there is an irreducible SFT $A \subseteq X_c$ such that $h(A) \geq h(X) - \epsilon$ and period$(A) = p$. Since

$$\liminf_n \frac{1}{np} \log q_{np}(X_c) \geq \liminf_n \frac{1}{np} \log q_{np}(A) = \lim_{n\to\infty} \frac{1}{np} \log q_{np}(A) = h(A),$$

the proof is complete. □

6. Sofic shift spaces have finite depth

We shall prove that a sofic shift space has finite depth by 'pruning' in the subset construction for a right-resolving presentation. We begin by introducing some convenient notation and terminology.

A factor map $\varphi : X \to Y$, with X a non-wandering SFT, will be called *component reduced* when there is no irreducible component $A \subseteq X$ such that $Y = \varphi(X \backslash A)$. If φ is not component reduced, we can delete irreducible components from X to obtain a new non-wandering SFT $X' \subseteq X$ and a factor map $\varphi|_{X'} : X' \to Y$ which is component reduced. We will refer to $\varphi|_{X'}$ as a *component reduction* of φ.

Let $(G, \mathcal{L})$ be a labeled graph. There are then a sofic shift space Y and a factor map $\varphi : X_G \to Y$ obtained 'by reading the labels', i.e. φ is a one-block map from X_G onto Y. We say that $\varphi : X_G \to Y$ is a *presentation* of Y. In this setting $\varphi(R(X_G)) = R(Y)$. The shift space $R(X_G)$ is the edge shift defined by the subgraph of G obtained by deleting all edges which go between different communicating classes of states; cf. pp. 118-119 of [LM]. We will denote this subgraph by $R(G)$, so that $R(X_G) = X_{R(G)}$ and $\varphi : R(X_G) \to R(Y)$ is then the presentation of $R(Y)$

given by $\left(R(G), \mathcal{L}|_{R(G)}\right)$. We will say that G is *non-wandering* when $G = R(G)$, and that the presentation $\varphi : X_G \to Y$ given by $(G, \mathcal{L})$ is *non-wandering* when G is.

A word $w \in \mathbb{W}(Y)$ is said to be *magic* for φ when there is a $j \in \{1, 2, \dots, |w|\}$ and an edge e in G such that $e_j = e$ for all elements $e_1 e_2 \dots e_{|w|} \in \varphi^{-1}(w)$. We shall mainly be working with right-resolving presentations, and in this case one can always require that $j = |w|$. As is well known and easily seen, a magic word for φ is synchronizing for Y.

To describe the subset construction, let $(G, \mathcal{L})$ be a labeled graph and $\varphi : X_G \to Y$ the presentation of Y given by it. We define a new graph $G^{\#}$ whose vertex set is the set $2^{\mathcal{V}}$ of subsets of the vertex set $\mathcal{V}$ of G. Let $\mathcal{A}$ be the alphabet of Y, i.e., the labels of G. We draw an arrow labeled $a \in \mathcal{A}$ from a subset $F \in 2^{\mathcal{V}}$ to another subset $F' \in 2^{\mathcal{V}}$, when

$$F' = \{x \in \mathcal{V} : \text{ there is an edge labeled } a \text{ from an element of } F \text{ to } x\}.$$

We denote this new labeled graph by $\left(G^{\#}, \mathcal{L}^{\#}\right)$. The corresponding presentation $\varphi^{\#} : X_{G^{\#}} \to Y$ of Y is right-resolving, and hence finite-to-one. For each $k \in \{1, 2, \cdots, \#\mathcal{V}\}$, let $G^{\#}_k$ denote the subgraph of $G^{\#}$ obtained by erasing all vertices $F \in 2^{\mathcal{V}}$ for which $\#F \neq k$, together with all arrows to or from such a vertex. We assume now that the original presentation $\varphi : X_G \to Y$ is right-resolving. This has the consequence that $\varphi^{\#}\left(\bigcup_k X_{G^{\#}_k}\right) = Y$; in fact, $\varphi^{\#}\left(X_{G^{\#}_1}\right) = Y$. A cycle

$$F \to F_1 \to F_2 \to \cdots \to F_n \to F \tag{6.1}$$

in $\bigcup_k G^{\#}_k$ defines a permutation μ of the elements in F, determined by the condition that for every $x \in F$ there are elements $x_1 \in F_1, x_2 \in F_2, \dots, x_n \in F_n$, such that there is a path $x \to x_1 \to x_2 \to \cdots \to x_n \to \mu(x)$ in G with the same label as (6.1). We will say that the cycle (6.1) is *static* when this permutation is trivial, and call a subgraph $H \subseteq \bigcup_k G^{\#}_k$ *hereditary* when the following holds: When (6.1) is a cycle in $\bigcup_k G^{\#}_k$ such that $F \in H$, then the whole cycle (6.1) is in H. Since every cycle in $\bigcup_k G^{\#}_k$ can be prolonged to obtain a static cycle, it suffices to check this condition for static cycles.

The preceding definitions are put to work in the proof of the following result.

Proposition 6.1. *Let $\varphi : X_G \to Y$ be a right-resolving presentation of the sofic shift space Y. For each $k = 0, 1, 2, \dots,$ there is a non-wandering and hereditary subgraph $H_{k+1} \subseteq \bigcup_{j \geq k+1} G^{\#}_j$ such that*

$$R\left(\partial^k Y\right) = \varphi^{\#}(X_{H_{k+1}}).$$

Proof. We will prove this by induction in k. Note that $H_1 = R\left(\bigcup_k G^{\#}_k\right)$ is a non-wandering and hereditary subgraph of $\bigcup_k G^{\#}_k$. Since $R(Y) = \varphi^{\#}\left(X_{H_1}\right)$, this yields the $k = 0$ case of the proposition. To handle the induction step, let H_k be a non-wandering and hereditary subgraph of $\bigcup_{j \geq k} G^{\#}_k$ such that $R\left(\partial^{k-1} Y\right) = \varphi^{\#}\left(X_{H_k}\right)$. By removing redundant components from H_k we may assume that $\varphi^{\#} : X_{H_k} \to R\left(\partial^{k-1} Y\right)$ is component reduced. For each vertex M in H_k, we let $\mathcal{F}(M)$ denote the follower-set $\mathcal{F}(M) \in 2^{\mathbb{W}(Y)}$ defined by H_k; cf. p. 78 of [LM]. Let H'_{k+1} be the subgraph of $\bigcup_{j \geq k+1} G^{\#}_j$ obtained by removing all vertices except those F for which there are subsets N_1, N_2 in F, both of which are vertices in H_k

and $\mathcal{F}(N_1) \neq \mathcal{F}(N_2)$. Since $N_1 \neq N_2$, and both sets contain at least k elements, $H'_{k+1} \subseteq \bigcup_{j \geq k+1} G^{\#}_j$. To see that H'_{k+1} is hereditary, let

$$F_0 \to F_1 \to F_2 \to \cdots \to F_0 \tag{6.2}$$

be a static cycle in $G^{\#}_j$ for some $j \geq k+1$, such that $F_0 \in H'_{k+1}$. Then $F_i = \{x^i_1, x^i_2, \cdots, x^i_j\} \subseteq \mathcal{V}$, and there are paths

$$x^0_a \to x^1_a \to x^2_a \to \cdots \to x^0_a \tag{6.3}$$

in G for all $a = 1, 2, \dots, j$, all with the same label as (6.2). Since F_0 is a vertex in H'_{k+1}, there are subsets $N_1, N_2 \subseteq F_0$, both vertices in H_k, such that $\mathcal{F}(N_1) \neq \mathcal{F}(N_2)$. The paths from (6.3) give us then paths

$$\begin{aligned} N_1 = N^1_0 \to N^1_1 \to N^1_2 \to \cdots \to N^1_0 = N_1, \\ N_2 = N^2_0 \to N^2_1 \to N^2_2 \to \cdots \to N^2_0 = N_2 \end{aligned} \tag{6.4}$$

in $\bigcup_{j \geq k} G^{\#}_j$ with the same label as (6.2). Since $N_1, N_2 \in H_k$, both these paths are in H_k, because H_k is hereditary. Since $\mathcal{F}(N_1) \neq \mathcal{F}(N_2)$, we conclude that $\mathcal{F}\left(N^1_a\right) \neq \mathcal{F}\left(N^2_a\right)$ for all $a = 0, 1, 2, \dots,$ because H_k is right-resolving. It follows that the path (6.2) proceeds in H'_{k+1}, and we conclude that H'_{k+1} is a hereditary subgraph of $\bigcup_{j \geq k+1} G^{\#}_j$. Set $H_{k+1} = R\left(H'_{k+1}\right)$, and note that H_{k+1} is then a non-wandering and hereditary subgraph of $\bigcup_{j \geq k+1} G^{\#}_j$.

Let $w \in \mathbb{W}\left(\partial^k Y\right)$. There must then be two words in $\mathbb{W}(X_{H_k})$ labeled w which terminate in vertices with different follower sets, since otherwise w would be magic for the presentation of $R\left(\partial^{k-1} Y\right)$ given by the follower-set graph of H_k; cf. p. 73 of [LM]. But then w would be synchronizing for $R\left(\partial^{k-1} Y\right)$, which is impossible since $w \in \mathbb{W}\left(\partial^k Y\right)$. By taking the union of the vertices from $G^{\#}$ occurring in the paths defining these two words in $\mathbb{W}(X_{H_k})$, both labeled w and terminating in vertices with different follower sets, we obtain a word in H'_{k+1} labeled w, and we conclude therefore that $\mathbb{W}\left(\partial^k Y\right) \subseteq \mathbb{W}\left(\varphi^{\#}(X_{H'_{k+1}})\right)$. It follows that $\partial^k Y \subseteq \varphi^{\#}\left(X_{H'_{k+1}}\right)$. Since $R\left(\varphi^{\#}\left(X_{H'_{k+1}}\right)\right) = \varphi^{\#}\left(X_{H_{k+1}}\right)$, it follows that

$$R\left(\partial^k Y\right) \subseteq \varphi^{\#}\left(X_{H_{k+1}}\right). \tag{6.5}$$

Next let $w = \varphi^{\#}(u)$, where $u \in \mathbb{W}\left(X_{H_{k+1}}\right)$. We claim that there are words w_1, w_2 in X_{H_k}, both labeled w, which terminate in vertices v_1 and v_2, respectively, with $\mathcal{F}(v_1) \neq \mathcal{F}(v_2)$. To see this, note that there is a periodic word $p \in X_{H_{k+1}}$ such that $u \subseteq p$. There must therefore be a static cycle (6.2) in H_{k+1} such that u is a subword of the label of this cycle. If we construct the paths (6.4) as above, they define two words in X_{H_k}, and the appropriate subwords of them will give words w_1 and w_2 with the asserted property. Assume that $\mathcal{F}(v_2)$ contains a word b which is not in $\mathcal{F}(v_1)$ (if no such word exists, we interchange the roles of w_1 and w_2). Since H_k is non-wandering, the vertices of w_1 are contained in the same irreducible component A of H_k. Since $\varphi^{\#} : X_{H_k} \to R\left(\partial^{k-1} Y\right)$ is component reduced, $\varphi^{\#}(X_A)$ is not contained in $\varphi^{\#}\left(X_{H_k \setminus A}\right)$. There is therefore a word $x \in \mathbb{W}(X_A)$ such that $\varphi^{\#}(x) \notin \mathbb{W}\left(\varphi^{\#}\left(X_{H_k \setminus A}\right)\right)$. Since A is irreducible, there is a word $a_0 \in \mathbb{W}(X_A)$ such that $x a_0 w_1 \in \mathbb{W}(X_A)$. Since $\varphi^{\#}$ is right-resolving, there is a word $a_1 \in \mathbb{W}(X_A)$ with the property that the words in X_A labeled $\varphi^{\#}(a_1)$ terminate in vertices of A with the same follower-set; cf. Proposition 3.3.16 of [LM]. Since A is irreducible, we may assume that $a_1 x a_0 w_1 \in \mathbb{W}(X_A)$. Set $a = \varphi^{\#}(a_1 x a_0)$ and note that $aw = \varphi^{\#}(a_1 x a_0 w_1) \in \mathbb{W}\left(R\left(\partial^{k-1}Y\right)\right)$. Since $b \in \mathcal{F}(w_2)$, $wb \in \mathbb{W}\left(R\left(\partial^{k-1}Y\right)\right)$. We claim that $awb \notin \mathbb{W}\left(R\left(\partial^{k-1}Y\right)\right)$. Indeed, if $d \in \mathbb{W}(X_{H_k})$ is a word such that $\varphi^{\#}(d) = awb$, d must be a word in X_A since $x \subseteq a$. By the choice of a_1, the $|a_1|$th edge in d must terminate in a vertex with the same follower-set as the terminal vertex of a_1. Since $\varphi^{\#}$ is right-resolving, it follows that the terminal vertex of the jth vertex in d must have the same follower-set as the terminal vertex of the jth edge in $a_1 x a_0 w_1$ when $|a_1| \leq j \leq |a_1 x a_0 w_1|$. In particular, when $j = |a_1 x a_0 w_1|$, this implies that b must be in the follower-set of the terminal vertex of w_1, since $\varphi^{\#}(d) = awb$. But this is not the case, and hence d cannot exist. Consequently, w cannot be synchronizing for $R\left(\partial^{k-1}Y\right)$. This proves that $\varphi^{\#}\left(X_{H_{k+1}}\right) \subseteq \partial^k Y$. Since H_{k+1} is non-wandering, we have in fact that $\varphi^{\#}\left(X_{H_{k+1}}\right) \subseteq R\left(\partial^k Y\right)$. Combined with (6.5) this yields the desired identity. □

It follows straightforwardly from Proposition 6.1 that $R(\partial Y)$ is a sofic shift space and that Y has finite depth, with $\text{Depth}(Y) \leq \max_{y \in Y} \#\varphi^{-1}(y)$. But we want to improve these conclusions a little, and in the process we will obtain an effective way to find the derived shift space of a sofic shift space.

The following lemma follows actually from Proposition 9.5 a) in [J1], as we shall explain in Remark 6.4 below. It also appears, in the irreducible case, as Lemma 1.1 of [T].

Lemma 6.2. *Let Y be a sofic shift space and $\varphi : X \to R(Y)$ a presentation of $R(Y)$ given by the non-wandering labeled graph $(G, \mathcal{L})$. Assume that φ is right-resolving, follower-separated and component reduced. Then a word $w \in \mathbb{W}(R(Y))$ is synchronizing for $R(Y)$ if and only if it is magic for φ.*

Proof. The 'if' part is well known and easy. The 'only if' part follows from the same reasoning as in the last part of the proof of Proposition 6.1. □

Lemma 6.3. *If Y is a sofic shift space, then there is a non-wandering, right-resolving, follower-separated and component reduced presentation $\varphi : X_G \to R(Y)$ of $R(Y)$.*

Proof. Start with a right-resolving presentation $\psi : X \to Y$; cf. Theorem 3.3.2 of [LM]. Then $\psi(R(X)) = R(Y)$, and $\psi|_{R(X)} : R(X) \to R(Y)$ remains right-resolving. Then the presentation $FR(X) \to R(Y)$ obtained by passing to the merged graph of $R(X)$ will be both right-resolving and follower-separated by Lemma 3.3.8 of [LM]. Furthermore, $FR(X)$ is a factor of $R(X)$ and hence non-wandering. Finally, we pass to a component reduction of $FR(X) \to R(Y)$, if necessary. □

Remark 6.4. In [J1], N. Jonoska calls a presentation of a sofic shift space *synchronizing* when every vertex in the graph of the presentation is the terminal vertex of a path whose label is a magic word. Some of the arguments used to prove Lemma 6.2 or Proposition 6.1 above show that a non-wandering, right-resolving, follower-separated and component reduced presentation must be synchronizing in that sense. Hence Lemma 6.3, combined with the above-mentioned arguments, implies Theorem 6.12 of [J1]: Every non-wandering sofic shift admits a synchronizing right-resolving presentation. In essense our proof of this is the same as Jonoska's, but we avoid the use of maximal FTR languages. Through this connection to Jonoska's work we conclude that the presentation φ of Lemma 6.3 is unique up to conjugacy; cf. Theorem 5.5 of [J1]. We may therefore refer to it as the minimal right-resolving synchronizing presentation of $R(Y)$. However, we will stick to the sligthly longer, but more descriptive terminology which we have used so far. Note that the presentation is simply the Fischer cover when Y is irreducible.

Proposition 6.5. *Let Y be a sofic shift space and $\varphi : X_G \to R(Y)$ a non-wandering, right-resolving, follower-separated and component reduced presentation of $R(Y)$. Then*

$$\partial Y = \varphi^{\#}\left(X_{G_2^{\#}}\right).$$

Proof. $\supseteq$: Let $x \in \varphi^{\#}(X_{G_2^{\#}})$. We must show that x does not contain a synchronizing word for $R(Y)$. By Lemma 6.2 this is the same as a magic word for φ. If x contains a magic word m, there is an n such that $u_n = v_n$ for all $u, v \in \varphi^{-1}(x)$. This is impossible, because $x \in \varphi^{\#}\left(X_{G_2^{\#}}\right)$.

$\subseteq$: Let $z \in R(Y)\backslash\varphi^{\#}\left(X_{G_2^{\#}}\right)$. We must show that z contains a synchronizing word for $R(Y)$—or equivalently, by Lemma 6.2, a magic word for φ. Let $\varphi^{-1}(z) = \{v^1, v^2, \ldots, v^n\}$. Since $z \notin \varphi^{\#}\left(X_{G_2^{\#}}\right)$ and φ is right-resolving, for each pair $i, j \in \{1, 2, \ldots, n\}$ there must be an $N_{ij} \in \mathbb{N}$ so large that $v^i_n = v^j_n$ for all $n \geq N_{ij}$. Set $N = \max_{i,j} N_{ij}$. We claim that there is a $k > N$ such that $u_N = v_N$ for all $u, v \in \varphi^{-1}(z_{[-k,k[})$. Since this implies that $z_{[-k,k[}$ is magic for φ, we will then be done. So assume to reach a contradiction that for all $k > N$ there are elements $u^k, v^k \in X_G$ such that $u^k_{[-k,k[}, v^k_{[-k,k[} \in \varphi^{-1}(z_{[-k,k[})$ and $u^k_N \neq v^k_N$. By the pigeonhole principle there is a sequence $N < k_1 < k_2 < \cdots$ such that $(u^{k_i}_N, v^{k_i}_N) = (u^{k_1}_N, v^{k_1}_N)$ for all i. By passing to subsequences we may assume that $\{u^{k_i}\}$ and $\{v^{k_i}\}$ both converge, say to $u, v \in X_G$. Then $u, v \in \varphi^{-1}(z)$ and $v_N \neq u_N$, contradicting the choice of N. □

Theorem 6.6. *If Y is a sofic shift space, so is ∂Y.*

Proof. This is an immediate consequence of Lemma 6.3 and Proposition 6.5. □

It follows of course that all the derived shift spaces of Y are sofic shift spaces. To estimate the depth of a sofic shift space we follow [FFJ] and let $\mathrm{pm}\,(\varphi)$ denote the maximal number of pre-images under φ of a periodic point in Y, and refer to this number as the *periodic multiplicity* of φ. Since φ is right-resolving, $\mathrm{pm}\,(\varphi)$ is the same as the the *multiplicity* of φ; cf. Lemma 2.3 of [FFJ].

Theorem 6.7. *Let Y be a sofic shift space, and $\varphi : X_G \to Y$ a right-resolving presentation of Y. Then* $\mathrm{Depth}(Y) \leq \mathrm{pm}(\varphi) - 1$.

Proof. Note that $R\left(G^{\#}_{k+1}\right) = \emptyset$ when $k + 1 > \mathrm{pm}(\varphi)$. It follows therefore from Proposition 6.1 that $R(\partial^k Y) = \emptyset$ when $k \geq \mathrm{pm}(\varphi)$. Since $\partial^k Y$ is a sofic shift space by Theorem 6.6, this implies that $\partial^k Y = \emptyset$ when $k \geq \mathrm{pm}(\varphi)$. □

Remark 6.8. There are cases where the inequality of Theorem 6.7 is an equality; this is for example the case when φ is the standard presentation of the even shift. However, in general the depth of a sofic shift space can be strictly smaller than the minimum value of $\mathrm{pm}(\varphi) - 1$, where φ varies over *all* presentations of Y. To see this, consider the sofic shift space Y presented by the labeled graph in Figure 5.

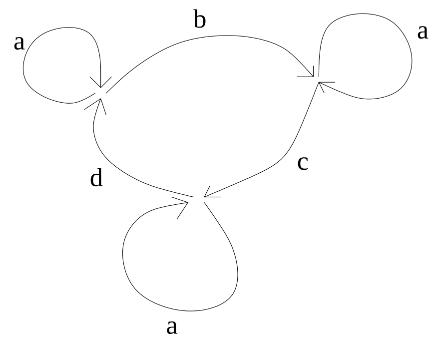


FIGURE 5.

If ψ is any presentation of Y, there is an irreducible presentation ψ_0 of Y, obtained from ψ by restriction to an irreducible component. Then $\mathrm{pm}(\psi) \geq \mathrm{pm}\,(\psi_0)$. Note that Y is an almost finite type shift since the graph in Figure 5 is also left-resolving, and hence ψ_0 must factor through the presentation given by the graph in Figure 5 according to [BKM]. Since the number of pre-images of a^∞ for the latter presentation is 3, we conclude that $\mathrm{pm}(\psi) - 1 \geq 2$. But it is easy to see that $\mathrm{Depth}(Y) = 1$ since ∂Y consists of the fixed point a^∞.

Remark 6.9. In a weak moment one might think that sofic shift spaces are characterized by the property of having finite depth or a finite number of irreducible components, at least among the synchronized systems. This is not the case; the context free shift is a synchronized system which is not sofic, but has depth 1 and only 3 irreducible components by Example 4.1.

Example 6.10. Proposition 6.5, Lemma 6.2 and Lemma 6.3 give a recipe for finding the derived shift space of a sofic shift space Y: Consider a right-resolving presentation of Y given by a labeled graph $(G, \mathcal{L})$. The restriction to $(R(G), \mathcal{L})$ is then a non-wandering right-resolving presentation of $R(Y)$. Note that $R(G)$ is then a disjoint union of finitely many irreducible labeled graphs. Throw away as many as possible of these, subject to the condition that the remaining graphs must still cover $R(Y)$. Finally, replace each of the remaining graphs by their follower-set graph. There is an algorithm for this; cf. pp. 92-94 of [LM]. Let H be the resulting graph. The associated presentation $\psi : X_H \to R(Y)$ is then non-wandering, right-resolving, follower-separated and component reduced. There is now an obvious algorithm for finding $H_2^{\#}$: Draw a graph whose vertices consist of two-elements sets $\{v_1, v_2\}$ of vertices from H, and draw an arrow labeled $a \in \mathcal{A}$ from $\{v_1, v_2\}$ to $\{v_1', v_2'\}$ when there are edges, both labeled a, from v_1 to $v'_{\mu(1)}$ and from v_2 to $v'_{\mu(2)}$ for some permutation μ of $\{1, 2\}$. The result is $H_2^{\#}$.

To illustrate the procedure consider the labeled graph G of Figure 6, which we borrowed with permission from p. 55 of [BK].

Let Y be the sofic shift space presented by G. The graph is clearly irreducible, right-resolving (biresolving, in fact), and by inspection one sees easily that it is also follower-separated. To determine the derived shift space ∂Y, we can go directly to the procedure described above to find $G_2^{\#}$. The result is shown in Figure 7.

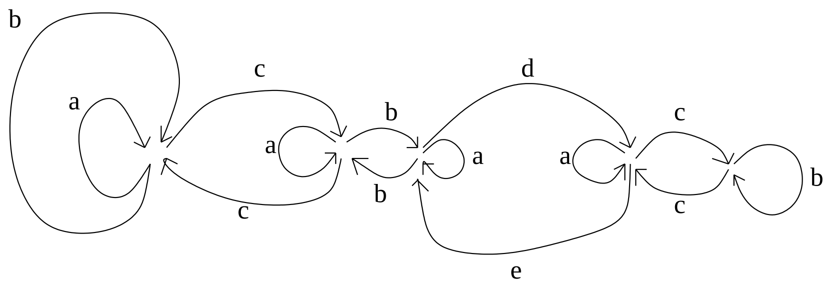


FIGURE 6. The graph G (reprinted from [BK]; print version © Springer-Verlag).

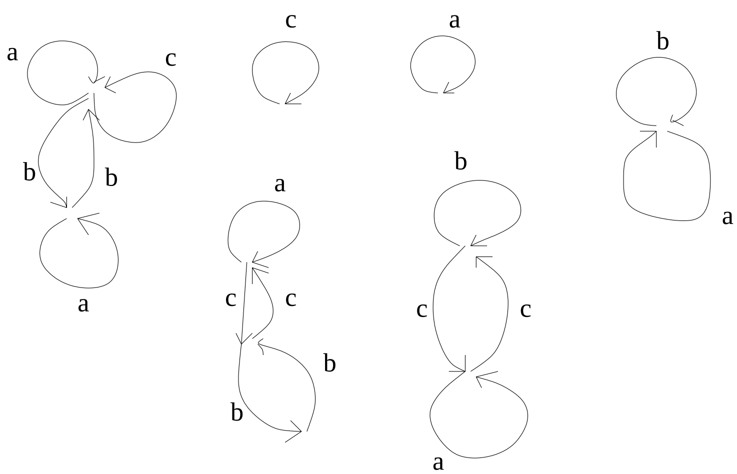


FIGURE 7. The graph $G_2^{\#}$.

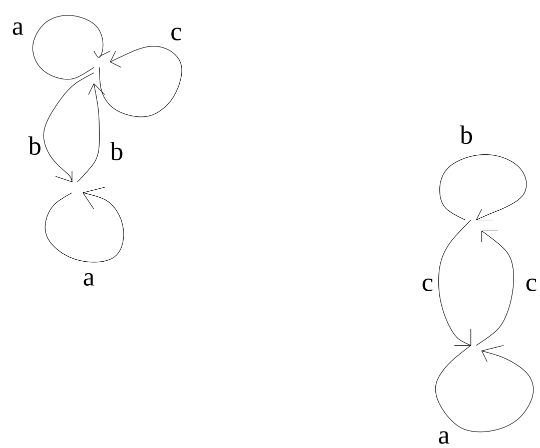


FIGURE 8. The graph H, presenting ∂Y; a component reduction of $G_2^{\#}$.

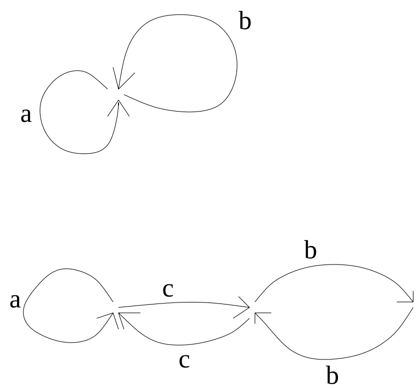


FIGURE 9. A presentation of $\partial^2 Y$; a component reduction of $H_2^{\#}$.

Although this graph does present ∂Y, it has a lot of redundancies, in the sense that a substantially smaller subgraph already covers ∂Y; namely the subgraph of Figure 8, which is a component reduction of $G_2^{\#}$.

Note that H is component reduced; the words *aca* and *cbc* are presented by a unique irreducible component of H. To check that H is also a follower-separated presentation of ∂Y we can check each component separately, and this is easy. We can therefore find $\partial^2 Y$ by constructing $H_2^{\#}$. Again the construction comes with redundancies which are removed by a component reduction. After that the result is the graph of Figure 9, which presents $\partial^2 Y$.

The graph in Figure 9 is non-wandering, right-resolving, follower-separated and component reduced, so we can apply Proposition 6.5 to find $\partial^3 Y$, which turns out to consist of the three fixed-points a^∞, b^∞ and c^∞. In particular, $\partial^3 Y$ is an SFT. Hence $\partial^4 Y = \emptyset$ and $\partial^3 Y$ is the floor of Y, i.e. $\operatorname{Depth} Y = 3$.

Proposition 6.11. *Let Y be a sofic shift space. Then* $\operatorname{Depth} Y = 0$ *if and only if* $R(Y)$ *is an SFT.*

Proof. If $R(Y)$ is an SFT, then every sufficiently long word in $R(Y)$ is synchronizing for $R(Y)$, and hence $\partial Y = \emptyset$, i.e. $\operatorname{Depth} Y = 0$. Conversely, assume that $\partial Y = \emptyset$. Then $\{y \in R(Y) : \ w \subseteq y \Rightarrow w \notin \mathbb{S}(Y)\}$ is empty, and a compactness argument shows that there must be an N such that all words in $\mathbb{W}(R(Y))$ of length $\geq N$ are synchronizing. It follows that $R(Y)$ is an SFT; cf. Theorem 2.1.8 of [LM]. $\square$

In particular, an irreducible sofic shift space is of finite type if and only if its depth is zero. There are strictly sofic shift spaces Y with $\operatorname{Depth} Y = 0$. A simple example is the sofic shift in Figure 4 of [J1]. By Proposition 6.11 all such examples must be wandering, i.e. $\overline{\operatorname{Per} Y} \neq Y$.

Remark 6.12. We show that for all $n = 1, 2, 3, \dots$ there is a mixing sofic shift space Y with $\operatorname{Depth} Y = n$ such that the floor $\partial^n Y$ is as general as possible; namely, an arbitrary sofic shift space Z for which $R(Z)$ is an SFT. For this purpose we prove by induction that for each $n = 0, 1, 2, \dots$ there is a right-resolving and follower-separated labeled graph G_n, which is irreducible and aperiodic when $n \geq 1$, such that the sofic shift Y_n presented by G_n has depth n and $\partial^n Y_n = Z$. To start the induction ($n = 0$), take G_0 to be any right-resolving and follower-separated

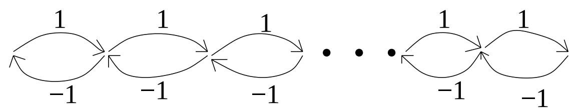


FIGURE 10.

presentation of Z. To handle the induction step, let G_n be a right-resolving and follower-separated labeled graph G_n such that the sofic shift Y_n presented by $G_n = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ has depth n, and $\partial^n Y_n = Z$. We then construct G_{n+1} as follows. The vertices of G_{n+1} consist of the disjoint union $\mathcal{V} \sqcup \mathcal{V}$ of two copies of the vertex set $\mathcal{V}$ from G_n. Let $v_1, v_2 \in \mathcal{V} \sqcup \mathcal{V}$. Draw an edge labeled $a \in \mathcal{A}$ (= the alphabet of Y_n) from v_1 to v_2 when v_1 and v_2 lie in the same copy of $\mathcal{V}$ and there is an edge labeled a from v_1 to v_2 in G_n. The result is the disjoint union of G_n with itself. We add $2(\#\mathcal{V})^2$ edges to this picture: Add $2(\#\mathcal{V})^2$ new letters to $\mathcal{A}$, say $x_{v,w}$ and $y_{v,w}$, $v, w \in \mathcal{V}$. For each pair of vertices $v, w \in \mathcal{V} \sqcup \mathcal{V}$ such that v is in the first copy of $\mathcal{V}$ and w is in the second, we draw an edge labeled $x_{v,w}$ from v to w and an edge labeled $y_{w,v}$ from w to v. Let G_{n+1} be the resulting labeled graph. It is easy to see that G_{n+1} is irreducible, right-resolving and follower-separated. To ensure that it is also aperiodic, we add a loop at one vertex, and label it by a letter we haven't used before. (This is not necessary when $n > 1$.) If Y_{n+1} is the sofic shift presented by G_{n+1}, Proposition 6.5 tells us that $\partial Y_{n+1} = Y_n$. Hence $\mathrm{Depth}\,(Y_{n+1}) = \mathrm{Depth}\,(Y_n) + 1 = n + 1$ and $\partial^{n+1} Y_{n+1} = \partial^n Y_n = Z$.

Remark 6.13. The fact that the depth of a sofic shift space can be any natural number can also be neatly illustrated by a series of irreducible (but not mixing) sofic shift spaces considered by B. Marcus on page 369 of [M1] and in 2.2 a of [M2] and called *charge constrained shifts* in [M1] and in Example 1.2.7 of [LM]: The labeled graph of Figure 10, where there are n ones and n minus-ones, presents the subshift Y_n of $\{1, -1\}^{\mathbb{Z}}$ with the property that the sum of symbols in any block is bounded by n in absolute value. It is easy to see that $\partial Y_1 = \emptyset$ and $\partial Y_n = Y_{n-1}$. In particular, $\mathrm{Depth}\,(Y_n) = n - 1$.

Lemma 6.14. *Let Y be a sofic shift space. Let $\varphi : X_G \to R(Y)$ be a non-wandering, right-resolving, follower-separated and component reduced presentation of $R(Y)$. For every irreducible component $Y_{(\alpha,0)}$ at level 0 in Y, there is a unique irreducible component H of G such that $\varphi^{-1}(\alpha) \subseteq \mathbb{W}(X_H)$. This component H has the property that $\varphi\,(X_H) = \overline{Y_{(\alpha,0)}}$.*

Proof. Let $u \in \alpha$. Since u is magic for φ by Lemma 6.2, the words in $\varphi^{-1}(u)$ must terminate at the same vertex. Since G is non-wandering, this implies that all paths in $\varphi^{-1}(u)$ run in the same irreducible component H of G. For any other element $v \in \alpha$ there is a word $x \in \mathbb{W}(R(Y))$ such that $uxv \in \mathbb{W}(R(Y))$. This implies that the common terminal vertex for the paths in $\varphi^{-1}(v)$ must also be in H, proving that $\varphi^{-1}(\alpha) \subseteq \mathbb{W}(X_H)$. Since every element $x \in Y_{(\alpha,0)}$ contains an element of α, it follows that $\varphi^{-1}(x) \in X_H$, and we conclude that $\varphi\,(X_H) \supseteq \overline{Y_{(\alpha,0)}}$. Let $z \in \overline{Y_{(\alpha,0)}}$ be an element with a dense orbit in $\overline{Y_{(\alpha,0)}}$. Then $z = \varphi(y)$ for some $y \in X_H$, and it follows that the orbit of z is also in $\varphi(X_H)$. Thus $\varphi(X_H) = \overline{Y_{(\alpha,0)}}$. In particular, $\varphi\,(X_H) = \varphi\,(X_{H_1})$ if $H_1 \subseteq G$ is another irreducible component such that $\varphi^{-1}(\alpha) \subseteq \mathbb{W}\,(X_{H_1})$. Since φ is component reduced, we conclude that $H = H_1$. $\square$

Remark 6.15. It follows from Lemma 6.14 that in a sofic shift space Y, a closed shift-invariant subset is the closure of an irreducible component at level 0 if and only if it is a maximal shift-invariant subset of $R(Y)$ with a dense orbit. Furthermore, it follows that every closed shift-invariant subset of $R(Y)$ with a dense orbit is contained in the closure of an irreducible component at level 0.

Proposition 6.16. *If $Y_{(\alpha,k)}$ is an irreducible component at level k in the sofic shift space Y, then $\overline{Y_{(\alpha,k)}}$ is an irreducible sofic shift space.*

Proof. Since $\partial^k Y$ is a sofic shift space by Theorem 6.6, we may assume that $k = 0$. Then the conclusion follows from Lemma 6.14. □

Remark 6.17. With the aid of Lemma 6.14 we can determine the irreducible components in a sofic shift space. To illustrate this we return to the example from Example 6.10. Besides the unique irreducible component at level 0 there are two irreducible components at level 1 in Y, each a dense subshift of one of the components in the graph of Figure 8. There are also two irreducible components at level 2, corresponding to the two components of the graph in Figure 9, and finally there are three components, all fixed points, on the floor at level 3. In principle one could describe an increasing sequence of irreducible SFTs in the sofic shift spaces presented by these graphs whose union is the corresponding irreducible component. For one of them, the full 2-shift in Figure 9, this is particularly simple since it's an irreducible SFT itself. However, this additional information will not be needed in the applications of the embedding and factor theorems which we prove in Section 8 and Section 9, respectively, so we refrain from doing it.

Note that Lemma 6.14 establishes a bijective correspondance between the irreducible components at level 0 in Y and the irreducible components in the non-wandering, right-resolving, follower-separated and component reduced presentation of $R(Y)$.

Proposition 6.18. *The number of components in a sofic shift space Y is finite.*

Proof. Since the derived shift spaces of Y are all sofic by Theorem 6.6 and since Y has finite depth by Theorem 6.7, it suffices to show that Y has only a finite number of irreducible components at level 0. This follows from Lemma 6.3 and Lemma 6.14. □

As shown by Example 4.7, there are non-sofic synchronized systems of finite depth with infinitely many irreducible components.

Lemma 6.19. *Let Y be a sofic shift space and $Y_{(\alpha,k)}$ an irreducible component at level k in Y, where $k \geq 1$. Then there is an irreducible component $Y_{(\beta,k-1)}$ at level $k-1$ in Y such that $\overline{Y_{(\alpha,k)}} \subseteq \overline{Y_{(\beta,k-1)}}$.*

Proof. This follows from Lemma 6.14, since $\overline{Y_{(\alpha,k)}}$ is a closed shift-invariant subset of $\partial^{k-1}Y$ and has a dense orbit; cf. Remark 6.15. □

Remark 6.20. The inclusion structure of closures of the irreducible components of a sofic shift space can be presented by a finite Bratteli diagram, where the irreducible components at level k are presented by dots and there is an edge between a dot at level $k-1$ and a dot at level k when the closure of the irreducible component at level k is contained in the closure of the irreducible component at level $k-1$. For

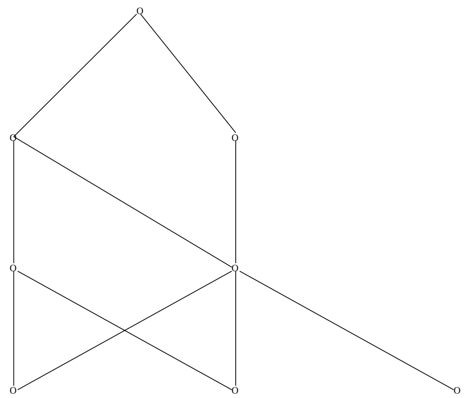


FIGURE 11. The inclusion pattern of the closures of the irreducible components in the sofic shift space of Figure 6.

the sofic shift space presented by Figure 6 above the result is the graph of Figure 11.

The following lemma makes it easy to find the periodic points in the top component of an irreducible sofic shift space, and to determine the period of its cyclic cover from the Fischer cover of the shift space.

Lemma 6.21. *Let Y be an irreducible sofic shift space with top component Y_c, and let $\psi : Y_G \to Y$ be its Fischer cover. Let $z \in \operatorname{Per} Y$. Then $z \in Y_c$ if and only if $\#\psi^{-1}(z) = 1$. Furthermore,* $\operatorname{period}(Y_c) = \operatorname{period}(Y_G)$.

Proof. If $\#\psi^{-1}(z) = 1$, $z \notin \partial Y$ by Proposition 6.5. Hence $z \in Y_c$ by Lemma 4.2. Conversely, when $z \in Y_c$ it follows from the definition of Y_c that $\#\psi^{-1}(z) = 1$ since a synchronizing word for Y is magic for ψ by Lemma 6.2.

It follows from the preceding that $\operatorname{period}(Y_G)$ divides $\operatorname{period}(Y_c)$. Set $p_0 = \operatorname{period}(Y_G)$ and let $y \in \operatorname{Per} Y_G$. Since Y_G is irreducible there is a cycle $q \in \mathbb{W}(Y_G)$ such that $\psi(q) \in \mathbb{W}(Y)$ is synchronizing for Y. Note that $(Y_G)^{p_0}$ is the disjoint union of mixing SFTs. For some $j \in \{0, 1, 2, \ldots, p_0 - 1\}$, $\sigma^j(q^\infty)$ and y are contained in the same of these components, say Z, of $(Y_G)^{p_0}$. Then $y_{[0,\operatorname{period}(y)[} \in \mathbb{W}(Z)$ and $\left(\sigma^j(q^\infty)\right)_{[0,2|q|[} \in \mathbb{W}(Z)$, and since σ^{p_0} is mixing on Z, there is an $N \in \mathbb{N}$ and for all $n, m \geq N$ there are words $a \in \mathbb{W}_{np_0}(Y_G)$ and $b \in \mathbb{W}_{mp_0}(Y_G)$ such that

$$y_{[0,\operatorname{period}(y)[}\, a \left(\sigma^j(q^\infty)\right)_{[0,2|q|[}\, b\, y_{[0,\operatorname{period}(y)[} \in \mathbb{W}(Y_G).$$

Note that $\psi\left(\left(\sigma^j(q^\infty)\right)_{[0,2|q|[}\right) \in \mathbb{W}(Y)$ is synchronizing for Y because it contains $\psi(q)$, and that

$$\left(y_{[0,\operatorname{period}(y)[}\, a \left(\sigma^j(q^\infty)\right)_{[0,2|q|[}\, b\right)^\infty$$

is a periodic point of period $2|q| + np_0 + mp_0 + \operatorname{period}(y)$. It follows that $M = \frac{2|q|+2Np_0+\operatorname{period}(y)}{p_0}$ has the property that for all $m \geq M$ there is a cycle r_m in Y_G such that $\psi(r_m)$ is synchronizing for Y and $|r_m| = mp_0$. This implies that for all $m \geq M$, Y_c contains a periodic point of period mp_0. If we choose $m \geq M$ such that m is relative prime to $\operatorname{period}(Y_c)$, we conclude that $\operatorname{period}(Y_c)$ divides $p_0 = \operatorname{period}(Y_G)$. ☐

To finish this section we show that the derived shift spaces and the irreducible components of an almost Markov sofic shift is again almost Markov, and that the affiliation pattern of almost Markov shift spaces are easy to determine from the appropriate covers.

Recall, [BK], that a shift space T is *almost Markov* when there is an SFT S and a biclosing factor map $\varphi : S \to T$. By work of Nasu, Kitchens, Boyle and Marcus, there is then a left- and right-resolving presentation, i.e. a biresolving presentation, of T; see [BKM]. An irreducible almost Markov shift space is often called an *almost finite type* shift space. An irreducible sofic shift space is an almost finite type shift space if and only if its Fischer cover is left-closing; cf. [N1]. It is this additional property of the Fischer cover we exploit in the following.

Proposition 6.22. *Let T be an almost Markov shift space. It follows that $\partial^k T$ is an almost Markov shift space for all k. Furthermore, $\partial T = \{x \in R(T) : \#\varphi^{-1}(x) \geq 2\}$ when $\varphi : X \to R(T)$ is a non-wandering, right-resolving, follower-separated and component reduced presentation of $R(T)$.*

Proof. It suffices to prove this for $k = 1$. Let $\varphi : S \to T$ be a biresolving presentation of T. Then $\varphi : R(S) \to R(T)$ is a biresolving presentation of $R(T)$. Let $Z_1, Z_2, \dots, Z_L$ be the components of $R(S)$ and note that each $\varphi(Z_i)$ is then an irreducible almost Markov shift space. For each i, let $\psi_i : X_{G_i} \to \varphi(Z_i)$ be the Fischer cover of $\varphi(Z_i)$. Let $G = G_1 \sqcup G_2 \sqcup \cdots \sqcup G_L$ be the disjoint union, and define $\psi : X_G \to R(T)$ by $\psi|_{X_{G_i}} = \psi_i$. By deleting the redundant G_i's we may assume that ψ is component reduced and hence follower-separated. It follows then from Proposition 6.5 that $\psi^{\#}(X_{G_2^{\#}}) = \partial T$. Since ψ_i is left-closing by [N1], it follows first that ψ is left-closing and then that $\psi^{\#} : X_{G_2^{\#}} \to \partial T$ is left-closing. Since the latter is also right-resolving, it is biclosing, and we conclude that ∂T is almost Markov, and that $\partial T = \{x \in R(T) : \#\psi^{-1}(x) \geq 2\}$. ☐

Corollary 6.23. *Let T be an almost Markov shift space, and T_c an irreducible component in T. Then $\overline{T_c}$ is almost of finite type.*

Proof. By Proposition 6.22 we can assume that T_c is at level 0. As in the proof of Proposition 6.22, we find that $R(T)$ is the union of a finite number, $T_1, T_2, \dots, T_k$, of subshifts almost of finite type, and that there is a non-wandering graph G with irreducible components $G_1, G_2, \dots, G_k$ and a right-resolving, follower-separated and component reduced presentation $\psi : X_G \to R(T)$ such that $\psi(X_{G_i}) = T_i, i = 1, 2, \dots, k$. It follows then from Lemma 6.14 that $\overline{T_c} = \psi(X_{G_i}) = T_i$ for some i. Hence $\overline{T_c}$ is almost of finite type. ☐

By Proposition 6.22, the derived shift space of an almost Markov shift T is exactly what is denoted by $fM(f)$ in [BK], i.e. the image in T of the multiplicity shift of the biresolving presentation of T of degree 1. In this way we may conclude from Proposition 2.5 of [BK] that any almost Markov shift occurs as the derived shift space of a mixing almost Markov shift. The identity between the derived shift space and the image of the multiplicity shift for the canonical biresolving cover of an irreducible almost Markov shift space T does not persist when the constructions are iterated; the second derived shift space is in general not the image in T of what is denoted by $M^{(2)}$ in [BK], or the image of M_3, for that matter. The example analyzed in Example 6.10 can serve as an illustration of this point. Furthermore,

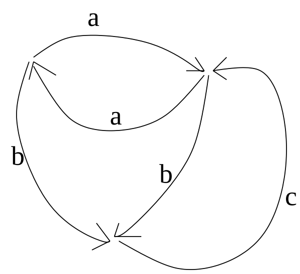


FIGURE 12.

in general the depth of T is not the same as the height of its canonical cover, as defined in [BK].

Proposition 6.24. *Assume that T is an almost finite type shift space, and let $\varphi : X_G \to R(T)$ be a non-wandering, right-resolving, follower-separated and component reduced presentation of $R(T)$. Let $p \in \mathbb{W}(T)$ be a minimal cycle and let $d \in \mathbb{N}$. Let T_c be an irreducible component at level 0 in T, and let $G_c \subseteq G$ be the irreducible component such that $\varphi^{-1}(c) \subseteq \mathbb{W}(X_{G_c})$; cf. Lemma 6.14. Then $p^\infty \in T_c^{(d)}$ if and only if $\varphi^{-1}(p^\infty) \cap X_{G_c}$ contains an element whose minimal period divides $|p|d$.*

Proof. Assume that p^∞ is d-affiliated to T_c. Then there are synchronizing words $w_1, w_2 \in c$ such that $w_1 p^{di} w_2 \in \mathbb{W}(R(T))$ for all $i \in \mathbb{N}$. Let v_0 be the terminal vertex of any element in $\varphi^{-1}(w_1)$, and note that $v_0 \in X_{G_c}$ by Lemma 6.14. Since w_1 is magic for φ by Lemma 6.2, this vertex is the same for all elements of $\varphi^{-1}(w_1)$. For fixed i the elements of $\varphi^{-1}(w_1 p^{id} w_2)$ must have $v_0 \in X_{G_c}$ as terminal vertex of the $|w_1|$th edge, and all the subsequent edges must be the same. In this way $\varphi^{-1}(w_1 p^{di} w_2)$ determines a path r_i in X_{G_c} of length $d|p|i$ with initial vertex v_0 such that $\varphi(r_i) = p^{di}$. Note that the first $|r_i|$ vertices occurring in r_{i+1} are exactly r_i, because φ is right-resolving. Note also that w_2 must be in the follower set of the terminal vertex of r_i for all i. Since there are only finitely many vertices, there must be $i, N \in \mathbb{N}$ such that the terminal vertex of r_i is the same as that of r_{i+N}. Write the last $Nd|p|$ edges of r_{i+N} as $a_1 a_2 \dots a_N$, where $|a_j| = d|p|$ and $\varphi(a_j) = p^d$ for all j. We claim that a_1 must be a cycle. So assume that a_1 is not a cycle. Then the terminal vertex v of a_1 must be different from the terminal vertex w of a_N. Furthermore, there are paths d_v and d_w, both labeled w_2, with initial vertices v and w. Then $(a_2 a_3 \dots a_N a_1)^\infty d_v$ and $(a_1 a_2 \dots a_N)^\infty d_w$ are different left-infinite paths with the same label. Since w_2 is magic for φ, their terminal vertices must be the same. This contradicts that $\varphi : X_{G_c} \to \varphi(X_{G_c})$ is left-closing by [N1], and we conclude that a_1 must be a cycle. Then $a_1^\infty \in X_{G_c}$ is a $d|p|$-periodic element of $\varphi^{-1}(p^\infty)$.

The converse holds for general sofic shifts by Lemma 7.7. □

Example 6.25. In the mixing sofic shift space presented by the graph shown in Figure 12, the fixed point a^∞ is 1-affiliated to the top component, but there is no fixed point in the subshift of the Fischer graph. Thus Proposition 6.24 fails in general.

7. Affiliation and morphisms between sofic shift spaces

Lemma 7.1. *Let $\varphi : X \to Y$ be an embedding of shift spaces. Assume that Y has finite depth. For each irreducible component $X_{(\alpha,j)}$ in X, there are a unique $j_\varphi \in \{0,1,2,\dots,\mathrm{Depth}(X)\}$ and a unique irreducible component $Y_{(\alpha_\varphi,j_\varphi)}$ in Y at level j_φ such that*

a) $\varphi\left(\overline{X_{(\alpha,j)}}\right) \subseteq \partial^{j_\varphi} Y$, *and*
b) $\varphi\left(X_{(\alpha,j)}\right) \cap Y_{(\alpha_\varphi,j_\varphi)} \neq \emptyset$.

$Y_{(\alpha_\varphi,j_\varphi)}$ has the property that

$$\varphi\left(X^{(d)}_{(\alpha,j)}\right) \subseteq Y^{(d)}_{(\alpha_\varphi,j_\varphi)}$$

for all $d \in \mathbb{N}$.

Proof. Since Y has finite depth, all periodic points of Y belong to an irreducible component by Lemma 4.2. Let j_φ be the least number in $\{0,1,2,\dots,\mathrm{Depth}(Y)\}$ with the property that there is an irreducible component at level j_φ containing a periodic point from $\varphi\left(X_{(\alpha,j)}\right)$. Then $\varphi\left(\mathrm{Per}\, X_{(\alpha,j)}\right) \subseteq \partial^{j_\varphi} Y$, and hence

$$\varphi\left(\overline{X_{(\alpha,j)}}\right) \subseteq \partial^{j_\varphi} Y, \tag{7.1}$$

since $\partial^{j_\varphi} Y$ is closed and $\overline{X_{(\alpha,j)}} = \overline{\mathrm{Per}\, X_{(\alpha,j)}}$ by Theorem 3.2. We claim that there is only one irreducible component at level j_φ in Y which contains a periodic point from $\varphi\left(X_{(\alpha,j)}\right)$. Indeed, if $Y_{(\beta,j_\varphi)}$ and $Y_{(\beta',j_\varphi)}$ are irreducible components at level j_φ and $x, y \in \mathrm{Per}\left(X_{(\alpha,j)}\right)$ are such that $\varphi(x) \in Y_{(\beta,j_\varphi)}$ and $\varphi(y) \in Y_{(\beta',j_\varphi)}$, then Theorem 3.2 tells us that there is an irreducible SFT $A \subseteq X_{(\alpha,j)}$ such that $x, y \in A$. Since $\varphi(x), \varphi(y) \in \varphi(A) \subseteq \partial^{j_\varphi} Y$ by (7.1), we conclude first from Lemma 5.3 that $\varphi(y)$ is 1-affiliated to $Y_{(\beta,j_\varphi)}$, and then from Lemma 5.4 that $\varphi(y) \in \overline{Y_{(\beta,j_\varphi)}}$. By Lemma 3.4 this is only possible if $\beta = \beta'$, proving the claim. Let $Y_{(\alpha_\varphi,j_\varphi)}$ be the irreducible component at level j_φ which contains a periodic element from $\varphi\left(X_{(\alpha,j)}\right)$. To prove that $Y_{(\alpha_\varphi,j_\varphi)}$ is the only irreducible component in Y satisfying a) and b), assume that $Y_{(\gamma,k)}$ also does. By applying Theorem 3.2 to both $X_{(\alpha,j)}$ and $Y_{(\gamma,k)}$ we conclude from b) that there are irreducible SFTs $A \subseteq X_{(\alpha,j)}, B \subseteq Y_{(\gamma,k)}$ such that $\varphi(A) \cap B \neq \emptyset$. Since $\varphi(A) \cap B$ is an SFT, it must contain a periodic point and we conclude that

$$\varphi\left(\mathrm{Per}\, X_{(\alpha,j)}\right) \cap Y_{(\gamma,k)} \neq \emptyset. \tag{7.2}$$

Hence $k \geq j_\varphi$. On the other hand, $k > j_\varphi$ is impossible, since $\varphi\left(\overline{X_{(\alpha,j)}}\right) \subseteq \partial^k Y$ and $\varphi\left(X_{(\alpha,j)}\right) \cap Y_{(\alpha_\varphi,j_\varphi)} \neq \emptyset$. Hence $k = j_\varphi$. As we have already shown that $Y_{(\alpha_\varphi,j_\varphi)}$ is the only irreducible component at level j_φ which contains a periodic point from $\varphi\left(X_{(\alpha,j)}\right)$, we conclude from (7.2) that $\gamma = \alpha_\varphi$.

To see that $\varphi\left(X^{(1)}_{(\alpha,j)}\right) \subseteq Y^{(1)}_{(\alpha_\varphi,j_\varphi)}$, let $x \in X^{(1)}_{(\alpha,j)}$. By definition of $(\alpha_\varphi, j_\varphi)$ there is a periodic point $y \in X_{(\alpha,j)}$ such that $\varphi(y) \in Y_{(\alpha_\varphi,j_\varphi)}$. By Lemma 5.7 there is an irreducible SFT $A \subseteq \overline{X_{(\alpha,j)}}$ such that $x, y \in A$. Then $\varphi(x), \varphi(y) \in \varphi(A) \subseteq \partial^{j_\varphi}(Y)$, and we conclude from Lemma 5.3 that $\varphi(x)$ is 1-affiliated to $Y_{(\alpha_\varphi,j_\varphi)}$. Consider finally a minimal cycle in $\mathbb{W}\left(\partial^j X\right)$ such that $x = p^\infty \in X^{(d)}_{(\alpha,j)}$, where $d \in \mathbb{N}$ is arbitrary. Let

$$X_{(\alpha,j)} = \bigsqcup_{\beta \in L} \left(X^{|p|d}\right)_{(\beta,j)}$$

be the decomposition (3.7) corresponding to $n = |p|d$. Then $x \in \left(X^{|p|d}\right)^{(1)}_{(\gamma,j)}$ for some $\gamma \in L$ by Lemma 5.5. From the first part of the proof we deduce that $\varphi(x) \in \left(Y^{|p|d}\right)^{(1)}_{(\gamma',j_\varphi)}$, where $\left(Y^{|p|d}\right)^{(1)}_{(\gamma',j_\varphi)}$ is the unique irreducible component in $Y^{|p|d}$ at level j_φ for which $\varphi\left(\operatorname{Per}\left(X^{|p|d}\right)_{(\gamma,j)}\right) \cap \left(Y^{|p|d}\right)_{(\gamma',j_\varphi)} \neq \emptyset$. Let

$$Y_{(\alpha_\varphi,j_\varphi)} = \bigsqcup_{\beta' \in L'} \left(Y^{|p|d}\right)_{(\beta',j_\varphi)}$$

be the decomposition (3.7) corresponding to $n = |p|d$. By definition of $(\alpha_\varphi, j_\varphi)$, $\varphi\left(\operatorname{Per} X_{(\alpha,j)}\right) \cap Y_{(\alpha_\varphi,j_\varphi)} \neq \emptyset$. There are therefore $\mu \in L$ and $\beta' \in L'$ such that $\varphi\left(\operatorname{Per} X^{|p|d}_{(\mu,j)}\right) \cap \left(Y^{|p|d}\right)_{(\beta',j_\varphi)} \neq \emptyset$. Since the shift induces a transitive action on L by Lemma 3.8, we may assume, by changing β' if necessary, that $\mu = \gamma$. It follows then that $\beta' = \gamma'$, i.e. $\varphi(x)$ is 1-affiliated to $\left(Y^{|p|d}\right)_{(\beta',j_\varphi)}$ in $Y^{|p|d}$, where $\beta' \in L'$. Since $\operatorname{period}(\varphi(x)) = |p|$, this means that there is a minimal cycle q of lenght $|p|$ such that $\varphi(x) = q^\infty$ and there are words $w_1, w_2 \in \alpha_\varphi \cap \mathbb{W}\left(\partial^{j_\varphi} Y^{|p|d}\right)$ such that $w_1 q^{di} w_2 \in \mathbb{W}\left(\partial^{j_\varphi} Y^{|p|d}\right)$ for all $i \in \mathbb{N}$. It follows that $\varphi(x) \in Y^{(d)}_{(\alpha_\varphi,j_\varphi)}$. □

It is notationally simpler to drop the explicit reference to the levels when handling the irreducible components, as we have done occasionally already. Let us therefore introduce the notation X_c for an irreducible component at any level in the shift space X. Then Lemma 7.1 tells us that the embedding φ induces a map $c \mapsto \varphi(c)$ from the irreducible components in X to the irreducible components in Y determined by the conditions that

1) $\varphi\left(\overline{X_c}\right) \subseteq \overline{X_{\varphi(c)}}$, and
2) $\varphi\left(X_c\right) \cap Y_{\varphi(c)} \neq \emptyset$.

This map has the property that

$$\varphi\left(X^{(d)}_c\right) \subseteq Y^{(d)}_{\varphi(c)}$$

for all $d \in \mathbb{N}$ and all c.

Lemma 7.2. *Let X and Y be irreducible sofic shift spaces and $\varphi : X \to Y$ a factor map. Let X_c and $Y_{c'}$ be the top components of X and Y. Let $x \in \operatorname{Per} X, \operatorname{period}(x) = n$, and $\operatorname{period}(\varphi(x)) = m$. Then*

$$x \in X^{(d)}_c \Rightarrow \varphi(x) \in Y^{\left(\frac{dn}{m}\right)}_{c'}.$$

Proof. Note first that if φ is a conjugacy rather than merely a factor map, we would have that $\varphi\left(X^{(d)}_c\right) = Y^{(d)}_{c'}$ by Proposition 5.6. We may therefore use a recoding to turn φ into a one-block map, and will therefore assume that this is the case. Let $\pi : X_G \to X$ be the Fischer cover of X and set $\psi = \varphi \circ \pi$. Let $p \in \mathbb{W}(X)$ be a minimal cycle of length n such that $p^\infty = x \in X^{(d)}_c$. Then there are elements $w, u \in c$ such that $wp^{di}u \in \mathbb{W}(X)$ for all $i \in \mathbb{N}$. Choose $w_1 \in \pi^{-1}(w)$ and set $\pi^{-1}(u) = \{u_1, u_2, \ldots, u_L\}$. Since u is magic for π, all u_i's terminate at the same vertex v_2 in G. Let v_1 be the initial vertex of w_1, let $t \in c'$ and choose a word $t_0 \in \psi^{-1}(t)$. Since X_G is irreducible, there are words $a_+, a_- \in \mathbb{W}(X_G)$ such that $t_0 a_+$ terminates at v_1 and the initial vertex of $a_- t_0$ is v_2. Set $\pi^{-1}(p) = \{p_1, p_2, \ldots, p_M\}$.

For each $i \in \mathbb{N}$ there is a string $j_1, j_2, \dots, j_{di} \in \{1,2,\dots,M\}$ and an $l \in \{1,2,\dots,L\}$ such that

$$w_1 p_{j_1} p_{j_2} \dots p_{j_{di}} u_l \in \mathbb{W}(X_G)$$

and $\pi(w_1 p_{j_1} p_{j_2} \dots p_{j_{di}} u_l) = w p^{di} u$. It follows that

$$t_0 a_- w_1 p_{j_1} p_{j_2} \dots p_{j_{di}} u_l a_+ t_0 \in \mathbb{W}(X_G)$$

for all $i \in \mathbb{N}$. Hence

$$t\psi(a_- w_1)\varphi(p)^{di}\varphi(u)\psi(a_+)t = \psi(t_0 a_- w_1 p_{j_1} p_{j_2} \dots p_{di} u_l a_+ t_0) \in \mathbb{W}(Y).$$

Since $t\psi(a_- w_1), \varphi(u)\psi(a_+)t \in c'$, this shows that $\varphi(p^\infty) \in Y_{c'}^{\left(\frac{nd}{m}\right)}$. □

Lemma 7.3. *Let Y_c be an irreducible component in the sofic shift space Y. Let $\left(\overline{Y_c}\right)_0$ be the top component of the irreducible sofic shift space $\overline{Y_c}$. It follows that $Y_c^{(d)} = \left(\overline{Y_c}\right)_0^{(d)}$ for all $d \in \mathbb{N}$.*

Proof. Assume that Y_c is at level j in Y, i.e. that $Y_c \subseteq \partial^j Y \backslash \partial^{j+1} Y$. Let $p \in \mathbb{W}\left(\overline{Y_c}\right)$ be a minimal cycle such that p^∞ is d-affiliated to Y_c. (Recall that the assumption that $p \in \mathbb{W}\left(\overline{Y_c}\right)$ is no restriction by Lemma 5.4.) It follows that there are words $w_1, w_2 \in c$ such that $w_1 p^{di} w_2 \in \mathbb{W}\left(R\left(\partial^j Y\right)\right)$ for all $i \in \mathbb{N}$. Let $\pi : X_G \to R\left(\partial^j Y\right)$ be a non-wandering, right-resolving, follower-separated and component reduced presentation of $R\left(\partial^j Y\right)$. Then $\overline{Y_c} = \pi(X_H)$ for some irreducible component $H \subseteq G$ with the property that $\pi^{-1}(w_i) \in \mathbb{W}(X_H), i = 1,2$; cf. Lemma 6.14. Since w_1 and w_2 are magic for π by Lemma 6.2, they are also magic for $\pi : X_H \to \overline{Y_c}$ and hence synchronizing for $\overline{Y_c}$. To conclude that $p^\infty \in \left(\overline{Y_c}\right)_0^{(d)}$, it suffices therefore to show that $w_1 p^{di} w_2 \in \mathbb{W}(Y_c)$ for all $i \in \mathbb{N}$. To this end, observe that there is a $b \in \mathbb{W}\left(R\left(\partial^j Y\right)\right)$ such that $w_2 b w_1 \in \mathbb{W}\left(R\left(\partial^j Y\right)\right)$. This shows that for any i, $(w_1 p^{di} w_2 b)^\infty \in \mathbb{W}\left(R\left(\partial^j Y\right)\right)$. Since $(w_1 p^{di} w_2 b)^\infty \in Y_c$ by definition of Y_c, the desired conclusion follows. Conversely, assume that $p^\infty \in \left(\overline{Y_c}\right)_0^{(d)}$. Then there are synchronizing words for $\overline{Y_c}$, say w_1 and w_2, such that $w_1 p^{di} w_2 \in \mathbb{W}(Y_c)$ for all $i \in \mathbb{N}$. Then w_1 and w_2 are magic for $\pi : X_H \to \overline{Y_c}$ by Lemma 6.14, and since π is component reduced, there are words $a, b \in \mathbb{W}(Y_c)$ such that $aw_1, w_2 b \in \mathbb{W}(Y_c)$ are magic words for π and hence synchronizing for $R\left(\partial^j Y\right)$. It follows from Lemma 3.3 that aw_1 and $w_2 b$ are words in c, and since $aw_1 p^{di} w_2 b \in \mathbb{W}(Y_c) \subseteq \mathbb{W}\left(R\left(\partial^j Y\right)\right)$ for all $i \in \mathbb{N}$, we conclude that $p^\infty \in Y_c^{(d)}$. □

It should be remarked that it is not generally the case that the top component of $\overline{Y_c}$ is Y_c; the latter is always a subset of the former by Lemma 3.10, but not vice versa. The components at level 1 in the sofic shift space Y presented by Figure 6 can serve to illustrate this: If Y_c denotes the irreducible component of Y whose closure is presented by the component on the right in Figure 8, then the fixed point a^∞ is in $\left(\overline{Y_c}\right)_0$ since the letter a is obviously synchronizing for $\overline{Y_c}$. However, $a^\infty \in \partial^2 Y$, and is therefore not contained in any irreducible component at level 1. In particular, it is not contained in Y_c. By considering the irreducible component whose closure is represented by the upper component in Figure 9, we see that this phenomenon can occur even when $\overline{Y_c}$ is an SFT.

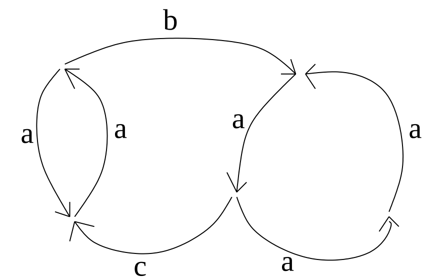


FIGURE 13.

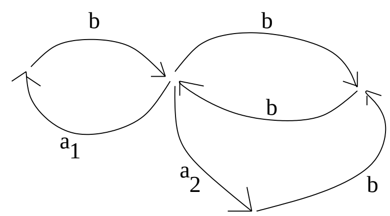


FIGURE 14.

Theorem 7.4. *Let X and Y be sofic shift spaces, and $\psi : X \to Y$ a morphism of shift spaces. It follows that there is a map $X_c \to Y_{\psi(c)}$ from the set of irreducible components in X to the set of irreducible components in Y, such that*

$$x \in X_c^{(d)} \Rightarrow \psi(x) \in Y_{\psi(c)}^{\left(\frac{\operatorname{period}(x)d}{\operatorname{period}(\psi(x))}\right)}. \tag{7.3}$$

In particular, $\psi\left(\overline{X_c}\right) \subseteq \overline{Y_{\psi(c)}}$.

Proof. Let X_c be an irreducible component in X. Then $\overline{X_c}$ is an irreducible sofic shift space by Proposition 6.16, and it follows that so is $\psi\left(\overline{X_c}\right)$. Let $\left(\psi\left(\overline{X_c}\right)\right)_0$ be its top component. By Lemma 7.1 there is a unique irreducible component $Y_{\psi(c)}$ of Y such that $\psi\left(\overline{X_c}\right) \subseteq \overline{Y_{\psi(c)}}$ and $\left(\psi\left(\overline{X_c}\right)\right)_0 \cap Y_{\psi(c)} \neq \emptyset$. (7.3) follows by combining Lemma 7.2, Lemma 7.3 and Lemma 7.1. □

In terms of the affiliation numbers introduced in the introduction, (7.3) says that an affiliation number for x to X_c is also an affiliation number for $\psi(x)$ to $Y_{\psi(c)}$. For all we know, it may be that Theorem 7.4 holds more generally, e.g. for arbitrary shift spaces of finite depth.

We remark that it will follow from Corollary 7.9 below that there is a finite set F of natural numbers such that (7.3) holds for all $d \in \mathbb{N}$ if and only if it holds for all $d \in F$.

It is important to realize that the affiliation numbers of a periodic point to a given irreducible component do not form a semigroup in $\mathbb{N}$, unlike the periods of the point. To illustrate this, consider the mixing sofic shift space X presented by the graph in Figure 13.

The fixed point a^∞ is d-affiliated to the top component in X exactly when $d \in 2\mathbb{N}$ or $d \in 3\mathbb{N}$. This follows from Proposition 6.24. Since $\operatorname{period}(a^\infty) = 1$, $2\mathbb{N} \cup 3\mathbb{N}$ are the affiliation numbers of a^∞ to the top component of X. They do not form a semigroup. Nonetheless, it is of course possible to talk about minimal affiliation numbers to an irreducible component. In this example it would be the numbers 2 and 3.

Remark 7.5. Theorem 7.4 gives a general explanation for the obstructions to the existence of non-trivial morphisms between certain sofic shift spaces which were observed in [B] (and in [LM]). To illustrate this we consider Example 3.1 in [B]. The sofic shift T is presented by the graph shown in Figure 14.

As pointed out by Boyle, there are no non-trivial morphisms to T from any irreducible SFT with a fixed point. This follows also from Theorem 7.4. In fact, let A be an irreducible sofic shift space which has a fixed point 1-affiliated to the top component. Thus A could for example be an SFT, the even shift, the shift of Figure 6 or any inclusive sofic shift space, in the sense of [B] which has a fixed point. Theorem 7.4 tells us that any morphism $\psi : A \to T$ would have to choose one of the components in T—the image of the top component in A under the map of components induced by ψ—and map any periodic point in A to a periodic point in T which is d-affiliated to this component for some d, since every periodic point in A is k-affiliated to the top component in A for some k, by Lemma 7.8. Since the unique fixed point in T is not 1-affiliated to the top component in T, the map ψ cannot choose the top component. There is only one more component in T, namely the fixed point b^∞ which constitutes the floor of T at level 1. It follows that ψ must map A onto b^∞.

With a little more effort one can show that the fixed point b^∞ in T is only d-affiliated to the top component in T when d is even, and that the conclusion extends to all irreducible sofic shift spaces A with a fixed point which is d-affiliated to the top component for some odd d.

Remark 7.6. Theorem 7.4 has also a bearing on the intrinsic multiplicity, $\mathrm{m}(Y)$, and the intrinsic periodic multiplicity, $\mathrm{pm}(Y)$, of an irreducible sofic shift space Y, which were introduced and studied in [FFJ]. Namely, it follows from Theorem 7.4 that when Y_c is the top component of Y we have

$$\#\pi^{-1}(y) \geq \min\{d \in \mathbb{N} : \ y \in Y_c^{(d)}\}$$

for every periodic point $y \in \operatorname{Per} Y$ and every factor map π mapping an irreducible SFT onto Y. In this way the affiliation pattern leads to a lower bound for $\mathrm{m}(Y)$ and $\mathrm{pm}(Y)$, which generally cannot be improved.

Lemma 7.7. *Let Y be a sofic shift space, and $\varphi : X_G \to R(Y)$ a non-wandering, right-resolving, follower-separated and component reduced presentation of $R(Y)$. Let $p \in \mathbb{W}(Y)$ be a minimal cycle and let $d \in \mathbb{N}$. Let $Y_{(\alpha,0)}$ be an irreducible component at level 0 in Y, and let $H \subseteq G$ be the irreducible component such that $\varphi^{-1}(\alpha) \subseteq \mathbb{W}(X_H)$; cf. Lemma 6.14. If $\varphi^{-1}(p^\infty) \cap X_H$ contains an element whose minimal period divides $|p|d$, then $p^\infty \in Y^{(d)}_{(\alpha,0)}$.*

Proof. Let q be a cycle in $\mathbb{W}(X_H)$ of length $d|p|$ such that $\varphi(q) = p^d$. Choose $a \in \mathbb{W}(X_H)$ such that $\varphi(a) \in \alpha$, and let $b_1, b_2 \in \mathbb{W}(X_H)$ be words such that the terminal vertex of ab_1 and the initial vertex of b_2a are both the initial vertex of q. Then $w_1 = \varphi(ab_1)$ and $w_2 = \varphi(b_2a)$ are words in α such that $w_1p^{id}w_2 = \varphi(ab_1q^ib_2a) \in \mathbb{W}(R(Y))$ for all $i \in \mathbb{N}$. Thus $p^\infty \in Y^{(d)}_{(\alpha,0)}$. □

Lemma 7.8. *Let $X_{(\alpha,k)}$ be an irreducible component at level k in the sofic shift space X. Let $K = \mathrm{pm}(\varphi)$ be the periodic multiplicity of φ, where $\varphi : X_G \to R\left(\partial^k X\right)$*

is a non-wandering, right-resolving, follower-separated and component reduced presentation. Then

a) $\operatorname{Per}\overline{X_{(\alpha,k)}} = \bigcup_{d\in\mathbb{N}} X_{(\alpha,k)}^{(d)} = \bigcup_{d=1}^{K} X_{(\alpha,k)}^{(d)} = X_{(\alpha,k)}^{(K!)}$, *and*

b) $x \in X_{(\alpha,k)}^{(d)} \;\Rightarrow\; \exists m \in \{1,2,\dots,\operatorname{pm}(\varphi)\}:\ m|d$ *and* $x \in X_{(\alpha,k)}^{(m)}$.

Proof. a) Note that $\bigcup_{d\in\mathbb{N}} X_{(\alpha,k)}^{(d)} \subseteq \operatorname{Per}\overline{X_{(\alpha,k)}}$ by Lemma 5.4. Since $\bigcup_{d=1}^{K} X_{(\alpha,k)}^{(d)} \subseteq X_{(\alpha,k)}^{(K!)}$, it suffices to show that $\operatorname{Per}\overline{X_{(\alpha,k)}} \subseteq \bigcup_{d=1}^{K} X_{(\alpha,k)}^{(d)}$. Let $p \in \overline{X_{(\alpha,k)}}$. Then $p = \varphi(q)$ for some $q \in \operatorname{Per} X_G$. By Lemma 7.7 this implies that $p \in X_{(\alpha,k)}^{(j)}$, where $j = \frac{\operatorname{period}(q)}{\operatorname{period}(p)}$. Since $j \leq \operatorname{pm}(\varphi)$, the proof is complete.

b) Let $q \in \mathbb{W}\left(\partial^k X\right)$ be the minimal cycle such that $x = q^\infty$ and $q = x_{[0,|q|[}$. Since $x \in X_{(\alpha,k)}^{(d)}$, there are words $w, u \in \alpha$ such that $wq^{di}u \in \mathbb{W}\left(R\left(\partial^k X\right)\right)$ for all $i \in \mathbb{N}$. Since w is magic for φ by Lemma 6.2, since φ is right-resolving and finally since there are only finitely many vertices in G, it follows that there is a $z \in \varphi^{-1}(x)$ such that

$$\forall i \in \mathbb{N}\ \exists u_i \in \varphi^{-1}(u):\ z_{[0,|q|di[}u_i \in \mathbb{W}\left(X_G\right). \tag{7.4}$$

Note that $\operatorname{period}(z) = |q|a$ for some $a \in \{1,2,\dots,\operatorname{pm}(\varphi)\}$. It follows from (7.4) that

$$\forall i \in \mathbb{N}\ \exists u_i \in \varphi^{-1}(u):\ z_{[0,|q|mi[}u_i \in \mathbb{W}\left(X_G\right), \tag{7.5}$$

where m is the greatest common divisor of a and d. It follows from (7.5) and Lemma 6.14 that there is a $v \in \alpha$ such that $vq^{mi}u \in \mathbb{W}\left(R\left(\partial^k X\right)\right)$ for all $i \in \mathbb{N}$, proving that $x \in X_{(\alpha,k)}^{(m)}$. □

Corollary 7.9. *Let X be an irreducible sofic shift space with top component X_c. Let $\operatorname{pm}(\varphi)$ be the periodic multiplicity of the Fischer cover of X. For each $d \in \mathbb{N}$,*

$$X_c^{(d)} = \bigcup_{\{1\leq m\leq \operatorname{pm}(\varphi):\ m|d\}} X_c^{(m)}.$$

Proof. This follows from b) of Lemma 7.8. □

8. When one of the shift spaces is an irreducible SFT

In this section we present our results on embedding and factor maps between shift spaces of unequal entropy when one of the spaces involved is an irreducible SFT.

8.1. **The global periodicity structure.** The structure of the cyclic cover, and in particular the fact that it need not be a partition, gives severe restrictions on which SFTs a non-mixing synchronized system can map into. The basic obstruction of this nature is described in the following proposition.

Proposition 8.1. *Let X be a synchronized system which is not mixing, and let $D_0, D_1, \dots, D_{p-1}$ be the cyclic cover of X. Then the following are equivalent:*

1) $\exists q \in \{2,3,\dots,p\}$, $q|p$, *such that, when* $i,k \in \{0,1,\dots,q-1\}$, $i \neq k$,

$$\left(\bigcup_{j=0}^{\frac{p}{q}-1} D_{i+jq}\right) \cap \left(\bigcup_{j=0}^{\frac{p}{q}-1} D_{k+jq}\right) = \emptyset.$$

2) *There exists a non-mixing irreducible SFT Y such that $X \subseteq Y$.*
3) *There exist a non-mixing irreducible SFT Y and a morphism $X \to Y$.*

Proof. 1) $\Rightarrow$ 2): The dynamical system $\left(\bigcup_{j=0}^{\frac{p}{q}-1} D_{jq}, \sigma^q\right)$ is a shift space, i.e., it is embedded into a full shift. This embedding extends to an embedding of X into an irreducible SFT of period q in the obvious way.

2) $\Rightarrow$ 3) is trivial.

3) $\Rightarrow$ 1): Let Y be a non-mixing irreducible SFT and $\psi : X \to Y$ a morphism. Set $q = \operatorname{period}(Y)$ and let $C_0, C_1, \dots, C_{q-1}$ be the cyclic cover of Y. The C_i's are mutually disjoint since Y is an SFT. Since σ^{pq} acts transitively on each D_i, there is for each i a j such that $\psi(D_i) \subseteq C_j$. It follows that there is a partition $K_0 \sqcup K_1 \sqcup \dots \sqcup K_{q-1}$ of $\{0, 1, 2, \dots, p-1\}$ such that

$$\bigcup_{i \in K_j} D_i \subseteq \psi^{-1}(C_j), \ j = 0, 1, \dots, q-1.$$

Since $\sigma(C_j) = C_{j+1}$ modulo q, $\sigma(D_i) = D_{i+1}$ modulo p, and ψ is shift commuting, it follows that $q|p$. By renumbering the C_j's we can arrange that $D_0 \subseteq \psi^{-1}(C_0)$. Then

$$K_l = \{l + jq : j = 0, 1, \dots, \frac{p}{q} - 1\}$$

for all $l = 0, 1, 2, \dots, q-1$. In particular,

$$\left(\bigcup_{j=0}^{\frac{p}{q}-1} D_{i+jq}\right) \cap \left(\bigcup_{j=0}^{\frac{p}{q}-1} D_{k+jq}\right) = \emptyset,$$

since $\psi^{-1}(C_i) \cap \psi^{-1}(C_k) = \emptyset$ when $i \neq k$. □

It follows from Proposition 8.1 that a non-mixing synchronized system, with the property that each pair from the cyclic cover intersects non-trivially, does not admit any morphism into any irreducible SFT which is not mixing. This applies, for example, to the irreducible sofic shift space in Remark 3.7. It is this obstruction which is responsible for the failure of Krieger's embedding theorem when the target shift is an irreducible SFT, but the domain only sofic. See Example 3.5 in [B] or Example 9.10 below for this obstruction in action, and Theorem 8.5 for the proof that it is the only obstruction.

Lemma 8.2. *Let X and Y be synchronized systems with top components X_c and Y_c. Let $\pi : X \to Y$ be a factor map. It follows that $p = \operatorname{period}(Y_c)$ divides $q = \operatorname{period}(X_c)$, and we can number the cyclic covers $C_0, C_1, \dots, C_{p-1}$ of Y and $D_0, D_1, \dots, D_{q-1}$ of X so that*

$$\pi(D_{i+jp}) = C_i \tag{8.1}$$

for all $i = 0, 1, \dots, p-1$ and all $j = 0, 1, \dots, \frac{q}{p} - 1$.

Proof. First let $C_0, C_1, \dots, C_{p-1}$ and $D_0, D_1, \dots, D_{q-1}$ be an arbitrary numbering of the cyclic covers of Y and X, respectively. Since σ^{pq} acts transitively on D_i by iii) of Corollary 3.12, there is a $j \in \{0, 1, 2, \dots, p-1\}$ such that $\pi(D_i) \subseteq C_j$. In fact, we claim that there is a surjective function $\mu : \{0, 1, \dots, q-1\} \to \{0, 1, \dots, p-1\}$ such that

$$\pi(D_i) \subseteq C_{\mu(i)} \tag{8.2}$$

for all $i \in \{0, 1, \ldots, q-1\}$. Indeed, it cannot happen that

$$\pi(D_i) \subseteq C_k \cap C_l \tag{8.3}$$

when $k \neq l$. This is because $Y = \bigcup_{j=0}^{p-1} \pi(D_j)$, so that at least one of the closed sets $\pi(D_j)$, $j = 0, 1, \ldots, q-1$, must have non-empty interior. Since $\sigma(\pi(D_i)) = \pi(D_{i+1})$ modulo q, they must all have non-empty interior, and hence (8.3) is impossible by the property iv) in Corollary 3.12 of a cyclic cover. We can therefore define μ as the map such that (8.2) holds. Then μ is surjective because π is, and hence $K_i = \mu^{-1}(\{i\})$, $i = 0, 1, \ldots p-1$, is a partition of $\{0, 1, \ldots, q-1\}$. Since $\sigma(C_i) = C_{i+1} \mod p$, it follows that $\# K_i = \# K_0$ for all i, and we conclude that $q = (\# K_0)\, p$. If we renumber the C_i's such that $\mu(0) = 0$, we find that $K_i = \{i + jp : \ j = 0, 1, \ldots, \frac{q}{p} - 1\}$ for all $i = 0, 1, \ldots, p-1$. To see that (8.1) holds, observe that since $\pi(D_{i+jp}) \subseteq C_i$ has non-empty interior, and σ^{pq} is mixing on C_i, there is a point $x \in \pi(D_{i+jp})$ whose orbit under σ^{pq} is dense in C_i. If $y \in D_{i+jp}$ is any pre-image of x under π, the σ^{pq}-orbit of y is a subset of D_{i+jp} whose image under π is dense in C_i. Hence (8.1) holds. □

Proposition 8.3. *Let Y be a synchronized system, and X an irreducible SFT. Let $C_0, C_1, \ldots, C_{p-1}$ and $D_0, D_1, \ldots, D_{q-1}$ be the cyclic covers of Y and X, respectively. Then Y is a factor of X if and only if p divides q and (C_0, σ^p) is a factor of $\left(\bigcup_{j=0}^{\frac{q}{p}-1} D_{jp}, \sigma^p\right)$.*

Proof. The 'only if' part follows from Lemma 8.2. The 'if' part follows from a standard argument based on the fact that the D_i's are mutually disjoint since X is an SFT. □

8.2. **When the target is an irreducible SFT.**

Theorem 8.4. *Let X and Y be irreducible shift spaces, X a synchronized system and Y an SFT. Assume that $h_{syn}(X) > h(Y)$. Then Y is a factor of X if and only if the following hold:*

i) *When X contains a periodic point of minimal period n, there is an $m \in \mathbb{N}$ such that $m|n$ and Y contains a periodic point of minimal period m.*
ii) *When $q = \operatorname{period}(Y)$ and $D_0, D_1, \ldots, D_{p-1}$ is the cyclic cover of X, then $q|p$ and*

$$\left(\bigcup_{j=0}^{\frac{p}{q}-1} D_{i+jq}\right) \cap \left(\bigcup_{j=0}^{\frac{p}{q}-1} D_{k+jq}\right) = \emptyset$$

when $i, k \in \{0, 1, \ldots, q-1\}$, $i \neq k$.

Proof. The necessity of condition i) is obvious and well known. The necessity of ii) follows from the argument used to prove 3) ⇒ 1) in Proposition 8.1.

The proof of the sufficiency of the two conditions hinges on Boyle's work in [B]. Given condition ii), it suffices to show that the dynamical system

$$\left(\bigcup_{j=0}^{\frac{p}{q}-1} D_{jq}, \sigma^q\right) \tag{$*$}$$

factors onto an irreducible component of Y^q. To this end note first that $(*)$ is (conjugate to) a shift space, and that Theorem 3.2 implies that there is an increasing sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ of irreducible SFTs in $(*)$ such that $\lim_{n\to\infty} h(A_n) = qh_{syn}(X)$. In particular, there is an n such that $h(A_n) > qh(Y)$. It follows from condition i) that when n is the minimal period of a periodic point in $(*)$, there is an m such that $m|n$ and m is the minimal period of a periodic point in Y^q. Then Theorem 3.3 of [B] lets us conclude that $(*)$ factors onto any irreducible component of Y^q. □

Theorem 8.5. *Let X and Y be irreducible shift spaces, X a synchronized system and Y an SFT. Assume that $h(Y) > h(X)$. Then X embeds into Y if and only if the following hold:*

i) $q_n(X) \leq q_n(Y)$ *for all* $n \in \mathbb{N}$.

ii) *When* $q = \operatorname{period}(Y)$ *and* $D_0, D_1, \dots, D_{p-1}$ *is the cyclic cover of* X, *then* $q|p$ *and*

$$\left(\bigcup_{j=0}^{\frac{p}{q}-1} D_{i+jq}\right) \cap \left(\bigcup_{j=0}^{\frac{p}{q}-1} D_{k+jq}\right) = \emptyset$$

when $i, k \in \{0, 1, \dots, q-1\},\ i \neq k$.

Proof. The proof is the same as the previous one, except that one uses Krieger's embedding theorem instead of Theorem 3.3 of [B]. We leave the details to the reader. □

8.3. **When the domain is an irreducible SFT.**

Theorem 8.6. *Let X and Y be shift spaces, X an irreducible SFT and Y of finite depth.*

a) *If X embeds into Y, there is an irreducible component Y_c at some level in Y such that $X \subseteq \overline{Y_c}$, $h(X) \leq h_{syn}\left(\overline{Y_c}\right)$ and $q_n(X) \leq q_n\left(Y_c^{(1)}\right)$ for all $n \in \mathbb{N}$.*

b) *If there is an irreducible component Y_c at some level in Y such that $h(X) < h_{syn}\left(\overline{Y_c}\right)$ and $q_n(X) \leq q_n\left(Y_c^{(1)}\right)$ for all $n \in \mathbb{N}$, then X embeds into Y. In fact, $X \subseteq \overline{Y_c}$ in this case.*

Proof. a) X is its own top component, so if $X \subseteq Y$, it follows from Lemma 7.1 and Lemma 5.4 that there is an irreducible component Y_c of Y such that $X \subseteq \overline{Y_c}$ and $\operatorname{Per} X \subseteq Y_c^{(1)}$. Hence $h(X) \leq h_{syn}\left(\overline{Y_c}\right)$ by Theorem 3.2, and $q_n(X) \leq q_n\left(Y_c^{(1)}\right)$ for all $n \in \mathbb{N}$.

b) It follows from Lemma 5.7 that there is an increasing sequence $B_1 \subseteq B_2 \subseteq \dots$ of irreducible SFTs in $\overline{Y_c}$ such that $\lim_{k\to\infty} h(B_k) = h_{syn}\left(\overline{Y_c}\right)$, $\operatorname{period}(B_k) = \operatorname{period}(Y_c^{(1)})$ for all k, and $Y_c^{(1)} = \operatorname{Per}\left(\bigcup_k B_k\right)$. In particular, for some $L \in \mathbb{N}$ we have $h(B_l) > h(X)$ when $l \geq L$. Note that $q_n(X) = 0,\ n \notin \mathbb{N}\operatorname{period}(Y_c^{(1)})$, since $q_n(X) \leq q_n\left(Y_c^{(1)}\right)$ for all n. Since

$$\begin{aligned} h(X) &= \limsup_n \frac{1}{n} \log q_n(X) < h(B_L) \\ &= \lim_{n\to\infty} \frac{1}{n \operatorname{period}\left(Y_c^{(1)}\right)} \log q_{n \operatorname{period}\left(Y_c^{(1)}\right)}(B_L), \end{aligned}$$

there is an $N \in \mathbb{N}$ so large that $q_n(X) \leq q_n(B_L)$ for all $n \geq N$. It follows that $q_n(X) \leq q_n(B_l)$ for all $n \geq N$, $l \geq L$. If we choose l large enough, then $q_n(B_l) =$

$q_n\left(Y_c^{(1)}\right), n < N$. Then $q_n(X) \leq q_n(B_l)$ for all $n \in \mathbb{N}$, and $h(X) < h(B_l)$. It follows then from Krieger's embedding theorem, Theorem 10.1.1 of [LM], that X embeds into B_l. Hence X embeds into $\overline{Y_c}$. $\square$

This result is the best we can hope to achieve about the embedding of an irreducible SFT of lower entropy into a shift of finite depth or even a sofic shift, as long as we do not know the necessary and sufficient conditions to have an embedding (i.e. a conjugacy) when the two shifts have the same entropy. The problem of finding such conditions seems to be open, even when both shift spaces are mixing SFTs.

Thanks to Proposition 8.3, the question of when a given irreducible sofic shift space is a factor of an irreducible SFT of larger entropy has now been reduced to the case when the sofic shift space is mixing. This case will be covered by Theorem 9.13 below. We include the result here, and start with the following lemma.

Lemma 8.7. *Let Y be an irreducible sofic shift space with top component Y_c. Every affiliation number of a periodic point to Y_c is divisible by the period of Y_c, i.e.*

$$Q_m\left(Y_c^{\left(\frac{n}{m}\right)}\right) \neq \emptyset \;\Rightarrow\; n \in \mathbb{N}\,\mathrm{period}\,(Y_c)\,.$$

Proof. Let $\varphi : Y_G \to Y$ be the Fischer cover of Y. When $Q_m\left(Y_c^{\left(\frac{n}{m}\right)}\right) \neq \emptyset$, it follows as in the proof of Lemma 7.8 b) that there are a cycle $p \in \mathbb{W}(Y_G)$ of length $|p| = m \cdot l$, where $l|\frac{n}{m}$, and a synchronizing word $u \in \mathbb{S}(Y)$ such that $\forall i \in \mathbb{N}\ \exists w_i \in \varphi^{-1}(u):\ p^i w_i \in \mathbb{W}(Y_G)$. All elements of $\varphi^{-1}(u)$ terminate at the same vertex in G, say v. Let r be a path in G connecting v to the initial vertex of p. Then pv_1r and p^2v_2r are cycles in Y_G whose labels are synchronizing for Y. It follows that $\varphi\left((pw_1r)^\infty\right)$ and $\varphi\left((p^2w_2r)^\infty\right)$ are in Y_c. Thus period$\,(Y_c)$ divides both period$\left((pw_1r)^\infty\right)$ and period$\left((p^2w_2r)^\infty\right)$, and hence also $|p^2w_2r| - |pw_1r| = |p|$. Since $|p|$ divides n, we conclude that period$\,(Y_c)$ divides n. $\square$

Theorem 8.8. *Let X be an irreducible SFT and Y an irreducible sofic shift space with top component Y_c. Assume that $h(X) > h(Y)$. Then Y is a factor of X if and only if*

$$q_n(X) \neq 0 \;\Rightarrow\; \sum_{\{m\in\mathbb{N}:\ m|n\}} q_m\left(Y_c^{\left(\frac{n}{m}\right)}\right) \neq 0 \tag{8.4}$$

for all $n \in \mathbb{N}$.

Proof. The 'only if' part follows from Theorem 7.4. To prove the 'if'-part, let $C_0, C_1, \ldots, C_{p-1}$ and $D_0, D_1, \ldots, D_{q-1}$ be the cyclic covers of Y and X, respectively. It follows from (8.4) and Lemma 8.7 that period$\,(Y_c)$ divides period(X), i.e. $p|q$. As pointed out in Proposition 8.3, it suffices now to show that (C_0, σ^p) is a factor of $\left(\bigcup_{j=0}^{\frac{q}{p}-1} D_{jp}, \sigma^p\right)$. Since (C_0, σ^p) is a mixing sofic shift space, and $\left(\bigcup_{j=0}^{\frac{q}{p}-1} D_{jp}, \sigma^p\right)$ is an irreducible SFT, we can apply the criterion arising from Theorem 9.13. Let x be an element of $\bigcup_{j=0}^{\frac{q}{p}-1} D_{jp}$ of minimal σ^p-period l. Since the cyclic cover of X is a partition, we conclude that the minimal σ-period of x is lp. It follows therefore from (8.4) that $Q_m\left(Y_c^{\left(\frac{lp}{m}\right)}\right) \neq \emptyset$ for some $m \in \mathbb{N}$ dividing lp. Let a be a cycle of length m in $\mathbb{W}(Y)$ such that $a^\infty \in Q_m\left(Y_c^{\left(\frac{lp}{m}\right)}\right)$. There are then words $w, v \in \mathbb{S}(Y)$ such that

$$wa^{\frac{lpi}{m}}v \in \mathbb{W}(Y)$$

for all $i \in \mathbb{N}$. Since w and v are equivalent in $\mathbb{S}(Y)$, there is a periodic point q containing both w and v. It follows that we can prolong w and v to words $w', v' \in \mathbb{S}\left(Y^p\right)$ which are both contained in q and satisfy

$$w'a^{\frac{lpi}{m}}v' \in \mathbb{W}(Y)$$

for all $i \in \mathbb{N}$. The minimal σ^p-period of a^∞ is $m_0 = \frac{\operatorname{lcm}(m,p)}{p}$, which divides l. Thus

$$b = a^{\frac{\operatorname{lcm}(m,p)}{m}}$$

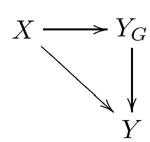
is a minimal σ^p-cycle in $\mathbb{W}\left(Y^p\right)$ such that

$$w'b^{\frac{li}{m_0}}v' \in \mathbb{W}\left(Y^p\right)$$

for all $i \in \mathbb{N}$. Since w' and v' are contained in the same periodic point, it follows that $a^\infty = b^\infty$ is $\frac{l}{m_0}$-affiliated to one of the p top components of Y^p. Since $\sigma(C_i) = C_{i+1}$ modulo p, we can combine Lemma 7.3 and Lemma 7.1 to conclude that for some $j \in \{0, 1, \dots, p-1\}$, $\sigma^j\left(a^\infty\right)$ is $\frac{l}{m_0}$-affiliated to the top component of C_0. Since the σ^p-period of $\sigma^j\left(a^\infty\right)$ is m_0, we can now conclude from Theorem 9.13 that (C_0, σ^p) is a factor of $\left(\bigcup_{j=0}^{\frac{q}{p}-1} D_{jp}, \sigma^p\right)$, as desired. □

In words, condition (8.4) says that a minimal period of a periodic point in X must be the affiliation number to the top component of a periodic point in Y.

By using Proposition 6.24 we can obtain an alternative proof of Theorem 8.8 in case Y is almost of finite type. This goes as follows: If Y is almost of finite type and is a factor of the irreducible SFT X, then there is a commuting diagram



of factor maps, where $Y_G \to Y$ is the Fischer cover of Y. This follows from [BKM]. As a consequence, $q_n(X) \neq 0 \ \Rightarrow \ \sum_{\{m \in \mathbb{N}:\ m|n\}} q_m\left(Y_G\right) \neq 0$, which by Lemma 7.7 implies that $\sum_{\{m \in \mathbb{N}:\ m|n\}} q_m\left(Y^{\left(\frac{n}{m}\right)}\right) \neq 0$. Conversely, if $\operatorname{Per} X \searrow \operatorname{Per} Y$, Proposition 6.24 shows that $q_n(X) \neq 0 \ \Rightarrow \ \sum_{\{m \in \mathbb{N}:\ m|n\}} q_m\left(Y_G\right) \neq 0$, so Boyle's result, Theorem 2.5 of [B], shows that Y_G is a factor of X. Hence Y is a factor of X.

9. Sofic shift spaces with transparent affiliation pattern

Definition 9.1. Let X and Y be synchronized systems with top components X_c and Y_c. We write $\operatorname{Per} X \hookrightarrow \operatorname{Per} Y$ when there are injective maps $\lambda_n : Q_n(X) \to Q_n(Y)$ such that

$$x \in X_c^{(d)} \Rightarrow \lambda_n(x) \in Y_c^{(d)}$$

for all d.

Note that it follows from Lemma 7.1 that a synchronized system X can only embed into a given shift space Y of finite depth when there is an irreducible component Y_c in Y with the property that $X \subseteq \overline{Y_c}$ and $\operatorname{Per} X \hookrightarrow \operatorname{Per} \overline{Y_c}$. In this section we identify a large class of mixing sofic shift spaces X for which the two conditions are almost also sufficient; cf. Theorem 9.9.

Through the rest of this section X is an irreducible sofic shift space and $\pi : X_G \to X$ its Fischer cover. Let $z \in \operatorname{Per} X$ be a periodic point of X, and consider a point $x \in X$. A *z-entry in x of length n* is an interval $x_{[i,j[}$ in x such that $j - i = n$, $x_{[i,j[} \subseteq z$ and $x_{[i-1,j[} \nsubseteq z$. Similarly, a *z-exit in x of length n* is an interval $x_{[i,j[}$ in x such that $j - i = n$, $x_{[i,j[} \subseteq z$ and $x_{[i,j+1[} \nsubseteq z$.

Definition 9.2. Let $z \in \operatorname{Per} X$. We say that z has *marked entries* when there is a natural number $N \in \mathbb{N}$ with the following property: When $x \in X$ and $x_{[i,j[}$ is a z-entry in x of length $2N$ or more, there is a unique $y \in \pi^{-1}(z)$ such that $w_{i+N-1} = y_{i+N-1}$ for all $w = w_{i-N} w_{i-N+1} \dots w_{i+N-1} \in \pi^{-1}\left(x_{[i-N,i+N[}\right)$.

We say that z has *marked exits* when there is a natural number $N \in \mathbb{N}$ with the following property: When $x \in X$ and $x_{[i,j[}$ is a z-exit in x of length $2N$ or more, there is a unique $y \in \pi^{-1}(z)$ such that $w_{j-N} = y_{j-N}$ for all $w = w_{j-N} w_{j-N+1} \dots w_{j+N-1} \in \pi^{-1}\left(x_{[j-N,j+N[}\right)$.

When $N \in \mathbb{N}$ satisfies both requirements in Definition 9.2, we will say that $2N$ is a *z-window.*

Lemma 9.3. *Let X be an irreducible sofic shift space and $\pi : X_G \to X$ the Fischer cover of X. A periodic point $z \in \operatorname{Per} X$ has marked entries and exits if and only if the following holds: When $\{z_n\}_{n=1}^{\infty} \subseteq X$ is a sequence such that $z_n \neq z$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} z_n = z$, then the set*

$$\overline{\bigcup_{n\in\mathbb{N}} \pi^{-1}(z_n)} \cap \pi^{-1}(z)$$

contains only one point.

Proof. Assume first that z has marked entries and exits, and consider a sequence $\{z_n\}_{n=1}^{\infty} \subseteq X$ such that $z_n \neq z$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} z_n = z$. Let $a, b \in \overline{\bigcup_{n\in\mathbb{N}} \pi^{-1}(z_n)} \cap \pi^{-1}(z)$ and let $l \in \mathbb{N}$ be larger than a z-window and also larger than $(M! + 1)\operatorname{period}(z)$, where M is the number of vertices in G. For each $k > l$ there are $x, y \in \bigcup_{n\in\mathbb{N}} \pi^{-1}(z_n)$ such that $a_{[-k,k[} = x_{[-k,k[}$, $b_{[-k,k[} = y_{[-k,k[}$. Since $\pi(x) \neq z$ and $\pi(y) \neq z$, this implies that $x_j = y_j, j \geq -k + l$, because z has marked entries and marked exits. It follows that $a_{[-k+l,k[} = b_{[-k+l,k[}$. Since $k > l$ is arbitrary, we conclude that $a = b$.

The converse can be proved by contradiction: If z fails to have either marked entries or marked exits, we can construct a sequence $\{z_n\}_{n=1}^{\infty} \subseteq X$ such that $z_n \neq z$ for all n, $\lim_{n\to\infty} z_n = z$, and such that $\overline{\bigcup_{n\in\mathbb{N}} \pi^{-1}(z_n)}$ contains at least two different elements from $\pi^{-1}(z)$. □

Definition 9.4. Let X be an irreducible sofic shift space. We will say that X has *transparent affiliation pattern* when all except finitely many periodic points in X are 1-affiliated to the top component, and the remaining periodic points have marked entries and marked exits.

It follows from Proposition 4.6, Lemma 9.3 and the uniqueness of the Fischer cover (cf. [K2], [K3]) that the property of having transparent affiliation pattern is invariant under conjugacies.

In [BK] Boyle and Krieger introduced a class of irreducible sofic shift spaces and called them *near Markov*. In our terminology an irreducible sofic shift space X is near Markov when it is almost of finite type and its derived shift space ∂X is a finite set. We want to show that near Markov shifts have transparent affiliation pattern.

Lemma 9.5. *If X is a near Markov shift, then there is an $N \in \mathbb{N}$ so large that the following holds: When $z \in \operatorname{Per} \partial X,\ x \in X$, and $x_{[i,j[} \subseteq z,\ x_{[i-1,j[} \nsubseteq z,\ j-i \geq N$, then there is one and only one $y \in \pi^{-1}(z)$ such that $\pi^{-1}\left(x_{[i-N,j[}\right) \subseteq \{ay_{[i,j[} : a \in \mathbb{W}_N\left(X_G\right)\}$.*

Proof. Let $m = \max_{z \in \operatorname{Per} \partial X} (V!+1) \operatorname{period}(z)$, where V is the number of vertices in the graph of the Fischer cover of X. As is well known and easily seen, each $y \in \pi^{-1}(z)$ is periodic and the elements of $\pi^{-1}(z)$ are fully separated in the sense that $y, y' \in \pi^{-1}(z),\ y_i = y'_i \Rightarrow y = y'$; cf. Exercise 9.1.3 of [LM]. By using this, and the fact that π is right-resolving, it follows that when $w = w_1 w_2 \dots w_J \in \mathbb{W}_J\left(X_G\right)$, $\pi(w) \subseteq z$ and $J \geq m$, there is a unique $y \in \pi^{-1}(z)$ such that $w_{[m,J[} \subseteq y$. This implies that

$$\pi^{-1}\left(x_{[i,j[}\right) \subseteq \bigcup_{y \in \pi^{-1}(z)} \{ay_{[i+m,j[} : a \in \mathbb{W}_m\left(X_G\right)\},$$

when $z \in \operatorname{Per} \partial X,\ x \in X$, and $x_{[i,j[} \subseteq z,\ j-i \geq m$. Since X is almost of finite type, a result of Nasu [N1] shows that π is also left-closing. As is well known, this implies that there is a $K \in \mathbb{N}$ such that when $k > K$ we have $w = w_1 w_2 \dots w_k,\ w' = w'_1 w'_2 \dots w'_k \in \mathbb{W}_k\left(X_G\right),\ \pi(w) = \pi(w'),\ w_k = w'_k \ \Rightarrow\ w_i = w'_i, \forall i > K$. Hence, if $N > \max\{m, K\}$, we have

$$\pi^{-1}\left(x_{[i-N,j[}\right) \subseteq \bigcup_{y \in \pi^{-1}(z)} \{ay_{[i,j[} : a \in \mathbb{W}_N\left(X_G\right)\}$$

when $z \in \operatorname{Per} \partial X,\ x \in X$, and $x_{[i,j[} \subseteq z,\ j-i \geq N$.

Let y and y' be two different elements of $\pi^{-1}(z)$. We claim that there is an $L \in \mathbb{N}$ so large that

$$\begin{gathered} w = w_{-L+1} w_{-L+2} \dots w_0,\ w' = w'_{-L+1} w_{-L+2} \dots w'_0 \in \mathbb{W}_L\left(X_G\right), \\ w_{-1} \neq y_{-1},\ w_0 = y_0,\ w'_0 = y'_0 \\ \Downarrow \\ \pi(w) \neq \pi(w'). \end{gathered} \tag{9.1}$$

To prove this by contradiction, assume that no such L exists. By compactness this gives us elements $a, a' \in X_G$ such that $\pi(a) = \pi(a'), a_i = y_i, a'_i = y'_i, i \geq 0$, while $a_{-1} \neq y_{-1}$. Since π is right-resolving and $a_i \neq a'_i, i \geq 0$, it follows that $a_i \neq a'_i$ for all $i \in \mathbb{Z}$. It follows then from Proposition 6.5 that $\pi(a) \in \partial X$. Since $\pi(a)$ is forward asymptotic to z, and ∂X is finite, it follows that $\pi(a)$ must be equal to z. This is impossible since $a \notin \pi^{-1}(z)$. Indeed, a is not even periodic.

Since $\pi^{-1}(\partial X)$ is finite, there is an $L \in \mathbb{N}$ such that (9.1) holds for all $z \in \partial X$ and all $y, y' \in \pi^{-1}(z)$. Let $N > \max\{m, K, L\}$. Then

$$\pi^{-1}\left(x_{[i-N,j[}\right) \subseteq \{ay_{[i,j[} : a \in \mathbb{W}_N\left(X_G\right)\}$$

for exactly one $y \in \pi^{-1}(z)$, when $z \in \operatorname{Per} \partial X$, $x \in X$, and $x_{[i,j[} \subseteq z$, $x_{[i-1,j[} \nsubseteq z$, $j - i \geq N$. ☐

A similar argument proves the following.

Lemma 9.6. *Assume that X is a near Markov shift. There is an $N \in \mathbb{N}$ so large that the following holds: When $z \in \operatorname{Per} \partial X$, $x \in X$, and $x_{[i,j[} \subseteq z$, $x_{[i,j+1[} \nsubseteq z$, $j-i \geq 2N$, then there is one and only one $y \in \pi^{-1}(z)$ such that $\pi^{-1}\left(x_{[i,j+N[}\right) \subseteq \{ay_{[i+N,j[}b : a, b \in \mathbb{W}_N\left(X_G\right)\}$.*

Proof. Left to the reader. ☐

The slight asymmetry between Lemma 9.5 and Lemma 9.6 arises because π is right-resolving, but only left-closing.

Proposition 9.7. *A near Markov shift space has transparent affiliation pattern.*

Proof. Since X has depth at most one, all periodic points outside ∂X are contained in the top component and are therefore also 1-affiliated to the top component. By Lemma 9.5 and Lemma 9.6 the remaining periodic points, all inside ∂X, have marked entries and marked exits. ☐

There are many other irreducible sofic shift spaces with transparent affiliation pattern. For example, an inclusive mixing sofic shift space, in the sense of [B], has transparent affiliation pattern. This includes the examples constructed in Remark 6.12, at least when the graph G_0 in the construction is irreducible. Hence mixing sofic shift spaces with transparent affiliation pattern can have arbitrarily large depth. But a mixing sofic shift space can easily have transparent affiliation pattern without being near Markov or having all periodic points 1-affiliated to the top component. The sofic shift considered in Remark 7.5 is an example which is not even almost of finite type. Other examples of irreducible sofic shift spaces with transparent affiliation pattern, but not almost of finite type, occur in Example 9.10 and Example 9.14 below.

9.1. **A version of Krieger's embedding theorem for sofic systems.**

Lemma 9.8. *Let $\pi : X_G \to X$ be the Fischer cover of the irreducible sofic shift space X. Then there is a number $R \in \mathbb{N}$ with the following property: When $p \in \mathbb{W}(X)$ is a cycle and $w, u \in \mathbb{W}(X)$ are two magic words for π such that $wp^iu \in \mathbb{W}(X)$ for all $i \in \mathbb{N}$, then there are words $w', u' \in \mathbb{W}(X)$ of length no more than R such that $w'p^iu' \in \mathbb{W}(X)$ for all $i \in \mathbb{N}$, and w', u' are both magic for π.*

Proof. Since G is irreducible, there is an $L \in \mathbb{N}$ so large that every pair of vertices in G can be connected by a path in G of length $\leq L$. Fix a word $s \in \mathbb{W}\left(X_G\right)$ such that $\pi(s)$ is magic for π.

Choose $w^1 \in \pi^{-1}(w)$ and set

$$\pi^{-1}(u) = \{u^1, u^2, \ldots, u^N\}, \qquad \pi^{-1}(p) = \{p_1, p_2, \ldots, p_M\}.$$

Since π is right-resolving, N is not larger than the number K of vertices in G. Since u is magic for π, all u^i's terminate at the same vertex v_2 in G. For each $i \in \mathbb{N}$ there are a string $j_1, j_2, \ldots, j_i \in \{1, 2, \ldots, M\}$ and an $l \in \{1, 2, \ldots, N\}$ such that

$$w^1 p_{j_1} p_{j_2} \ldots p_{j_i} u^l \in \mathbb{W}(X_G).$$

There are words $a_\pm \in \mathbb{W}(X_G)$ with $|a_\pm| \leq L$ such that the terminal vertex of $sa_- \in \mathbb{W}(X_G)$ is the terminal vertex of w^1, and the initial vertex of a_+s is v_2. Set $w' = \pi(sa_-)$. Since the u^i's are different and π is right-resolving, $u_0^i \neq u_0^j$ when $i \neq j$. If $u_{K!}^i \neq u_{K!}^j$ for all $i \neq j$, there are elements $a < b$ in $\{0, 1, 2, \ldots, K!\}$ such that $u_a^i = u_b^i$ for all $i = 1, 2, \ldots, N$. In this case, set

$$v_j^i = \begin{cases} u_j^i, & j = 0, 1, 2, \ldots, a, \\ u_{b+(j-a)}^i, & j > a. \end{cases}$$

Then the words $v^1, v^2, \ldots, v^N$ will have the same image under π, all of them will terminate at v_2, and the initial vertex of v^i will be the same as that of u^i. Note that $|v^i| \leq |u^i| - 1$ for all i. By repeating this process a finite number of times we obtain words $v^1, v^2, \ldots, v^N \in \mathbb{W}(X_G)$ such that $|v^i| \leq (K+1)!$, $\pi(v^i) = \pi(v^1)$, and v^i has the same initial vertex as u^i and terminal vertex v_2, for all i. Set $u' = \pi(v^i a_+ s)$, and note that $|u'| \leq R$, when $R = L + |s| + (K+1)!$. Since $w' p^i u' \in \mathbb{W}(X)$ for all i, the proof is complete. □

Theorem 9.9. *Let X and Y be sofic shift spaces, X mixing with transparent affiliation pattern. Assume that there is an irreducible component Y_c in Y, at some level, such that*

1) $h(X) < h\left(\overline{Y_c}\right)$, *and*
2) $\operatorname{Per} X \hookrightarrow \operatorname{Per} \overline{Y_c}$.

Then X embeds into $\overline{Y_c} \subseteq Y$.

Proof. It follows from assumption 2) that there is a shift-commuting embedding $\lambda : \operatorname{Per} X \to \operatorname{Per} \overline{Y_c}$ such that $\lambda\left(X_0^{(d)}\right) \subseteq Y_0^{(d)}$ for all $d \in \mathbb{N}$, where X_0 and Y_0 denote the top components of X and $\overline{Y_c}$, respectively. By Lemma 5.7 there is an increasing sequence of irreducible SFTs $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ in $\overline{Y_c}$ such that $\mathbb{W}(A_n)$ contains a synchronizing word for $\overline{Y_c}$, $\operatorname{period}(A_n) = \operatorname{period}(Y_0^{(1)})$ for all n, $\operatorname{Per} \bigcup_n A_n = Y_0^{(1)}$, and $\lim_{n\to\infty} h(A_n) = h\left(\overline{Y_c}\right)$. From this sequence we choose an irreducible SFT A_K such that $h(A_K) > h(X)$. We claim that there is an $N_1 \in \mathbb{N}$ such that

$$q_n\left(X_0^{(1)}\right) \leq q_n(A_K) \tag{9.2}$$

when $n \geq N_1$. To see this, note first that

$$\limsup_{n\to\infty} \frac{1}{n} \log q_n\left(X_0^{(1)}\right) \leq \limsup_{n\to\infty} \frac{1}{n} \log q_n(X) \leq h(X) \tag{9.3}$$

by Proposition 4.1.15 of [LM]. The existence of λ implies that $q_n\left(X_0^{(1)}\right) \leq q_n\left(Y_0^{(1)}\right)$ for all $n \in \mathbb{N}$, and hence that $\operatorname{period}(Y_0^{(1)})$ divides $\operatorname{period}(X_0^{(1)})$. Since X is mixing, $\operatorname{period}(X_0^{(1)}) = 1$ by Lemma 3.6, and hence $\operatorname{period}(Y_0^{(1)}) = 1$. Then $\operatorname{period}(A_n) = 1$, and A_n is mixing for each n. Since $\overline{Y_c}$ is the closure of the union of the A_n's, it follows that $\overline{Y_c}$ is itself mixing. Furthermore, $h(A_K) = \lim_{n\to\infty} \frac{1}{n} \log q_n(A_K)$, and in combination with (9.3) this implies (9.2). Let $\mathcal{P}$ be the periodic points of X that are not 1-affiliated to X_0. Since $\mathcal{P}$ is finite by assumption there is an $N_2 \geq N_1$ such that $Q_n(X) = Q_n\left(X_0^{(1)}\right)$ for all $n \geq N_2$. Thanks to (9.2) we can therefore redefine λ on $\bigcup_{n\geq N_2} Q_n(X)$ to arrange that

$\lambda(Q_n(X)) \subseteq A_K$ for all $n \geq N_2$. Since $\operatorname{Per}\bigcup_n A_n = Y_0^{(1)}$ and $\lambda\left(X_0^{(1)}\right) \subseteq Y_0^{(1)}$, we can increase K to ensure that

$$\lambda\left(X_0^{(1)}\right) \subseteq \operatorname{Per} A_K. \tag{9.4}$$

If $\overline{Y_c} = A_k$ for some k, we can finish the proof by appealing directly to Krieger's embedding theorem; cf. [K1] or Corollary 10.1.9 of [LM]. We can therefore assume that $A_k \neq A_{k+1}$ for all k. Being sofic and mixing, $\overline{Y_c}$ has a transition length. Let L be a natural number which is both a common transition length for $\overline{Y_c}$ and A_K, and a step-length for A_K. Let $d \in \mathbb{W}(A_K)$ be a word which is synchronizing for $\overline{Y_c}$. We set $D = 2L + |d|$. Set $P_0 = \max\{\operatorname{period}(y) : y \in \pi^{-1}(\mathcal{P})\}$, and let P be both larger than P_0 and larger than a z-window for all $z \in \mathcal{P}$.

Let R be the number from Lemma 9.8. Since $h(\overline{Y_c}) > h(A_K) = h(A_K \cup \lambda(\mathcal{P}))$, there are an integer $K' > K$ and a word $u_0 \in \mathbb{W}(A_{K'})$ such that $u_0^2 \in \mathbb{W}(A_{K'})$, u_0 is synchronizing for $A_{K'}$, and u_0 does not occur in $\mathbb{W}(A_K \cup \lambda(\mathcal{P}))$. We can construct a word $u \in \mathbb{W}(A_{K'})$ which ends and begins with u_0 and has no appearances of u_0 in between. Since $\mathbb{W}(A_K)$ contains a synchronizing word for $\overline{Y_c}$, we can arrange that u is synchronizing for $\overline{Y_c}$. Finally, we can also arrange that $|u| - 3|u_0| > 2(L+R+D)$. Then $u^2 \in \mathbb{W}(A_{K'})$, and u appears exactly twice in u^2.

Since $h(X) < h(A_K)$, there is a $M \in \mathbb{N}$ so large that

$$\# W_n(X) \leq \# W_{n-2L-2|u|}(A_K)$$

for all $n \geq M$. We may assume that $M \geq 2P + 2L + 3|u|$. Let $c_n : W_n(X) \to \# W_{n-2L-|u|}(A_K)$ and $c'_n : W_n(X) \to \# W_{n-2L-2|u|}(A_K)$ be injective maps, $n \geq M$. For each $w \in W_n(X)$ choose words $a_w, b_w, a'_w, b'_w \in \mathbb{W}_L(Y_c)$ such that $u a_w c_n(w) b_w u \in \mathbb{W}(Y_c)$ and $u a'_w c'_n(w) b'_w u \in \mathbb{W}(Y_c)$.

For each $z \in \operatorname{Per} X$ we choose a minimal cycle p_z such that $z = (p_z)^\infty$. We choose p_z such that $p_z = p_{z'}$ when z and z' are in the same orbit—this is crucial in order to insure that the map we are constructing will commute with the shift. Assume now that $z \in X_0^{(1)}$. Then $\lambda(z)$ is 1-affiliated to Y_0 and $\lambda(z) \in A_K$. Choose a $i_0 \in \mathbb{Z}$ such that $z_{[i_0, i_0+\operatorname{period}(z)[} = p_z$. We can choose words $u_z, v_z^0 \in \mathbb{W}(A_K)$ such that $u_z \lambda(z)_{[i_0,i_0+L[} v_z^0 \in \mathbb{W}(A_K)$, $d \subseteq u_z, d \subseteq v_z^0$ and $|u_z| = 2L + |d|$, $|v_z^0| = L + |d|$. Set $v_z = \lambda(z)_{[i_0,i_0+L[} v_z^0$ and note that

$$u_z \lambda(z)_{[i_0+i\operatorname{period}(z), i_0+j\operatorname{period}(z)[} v_z \in \mathbb{W}(A_K) \tag{9.5}$$

for all $i < j$ in $\mathbb{Z}$, since L is a step-length for A_K. Note that $|u_z| = |v_z| = D$ for all z, and that u_z, v_z are synchronizing for $\overline{Y_c}$. By prolonging u_z and v_z we obtain words $u_\pm(i) \in \mathbb{W}_i(A_K)$ such that

$$u_-(i) \lambda(z)_{[i_0+k\operatorname{period}(z), i_0+l\operatorname{period}(z)[} u_+(j) \in \mathbb{W}(A_K) \tag{9.6}$$

for all $k < l$ in $\mathbb{Z}$ and all $i, j \geq D$. Furthermore, we can choose words $e_\pm \in \mathbb{W}_L(Y_c)$ such that $u e_- u_-(i), u_+(i) e_+ u \in \mathbb{W}(Y_c)$. Then

$$u e_- u_-(i) \lambda(z)_{[i_0+k\operatorname{period}(z), i_0+l\operatorname{period}(z)[} u_+(j) e_+ u \in \mathbb{W}(Y_c) \tag{9.7}$$

for all $k < l$ in $\mathbb{Z}$ and all $i, j \geq D$. Observe that we can choose $u_\pm(i)$ and $e_\pm$ such that they only depend on $\lambda(z)_{[i_0,i_0+L[}$ and i.

Let $x \in X$. We are going to introduce 4 types of intervals which may occur in x:

1) the special long intervals,
2) the long marker intervals,

3) the moderate marker intervals, and
4) the remaining intervals.

The special long intervals. A *maximal $\mathcal{P}$-stretch* in x is a (possibly infinite) interval $I \subseteq \mathbb{Z}$ such that $\#I \geq 2P+1$, $x_{[i,j[} \subseteq z$ for some $z \in \mathcal{P}$ when $i,j \in I$ and $i<j$, and such that no interval in x with these properties contains I properly. It follows from Lemma 2.3 of [B] that $x_I \subseteq z$ for a unique $z \in \mathcal{P}$ when I is a maximal $\mathcal{P}$-stretch in x. By Lemma 2.3 of [B], different maximal $\mathcal{P}$-stretches overlap in at most $2P$ coordinates. Choose for each $y \in \pi^{-1}(\mathcal{P})$ a minimal cycle $c_y \in \mathbb{W}(X_G)$ such that $y = (c_y)^\infty$. We choose c_y such that $c_y = c_{y'}$ when y and y' are in the same orbit; this is important in order to ensure that the map we are constructing commutes with the shift. Let I be a maximal $\mathcal{P}$-stretch in x such that $\#I \geq 2P+2M+1$, and assume first that I is finite, say $I = [i,j[$. Since P is larger than a z-window, for all $z \in \mathcal{P}$ there is a unique $y \in \pi^{-1}(\mathcal{P})$ such that $w_{[i+P,j-P[} = y_{[i+P,j-P[}$ when $w = w_{i-P}w_{i-P+1}\dots w_{i+P-2}w_{j+P-1} \in \pi^{-1}\left(x_{[i-P,j+P[}\right)$. Set

$$i_0 = \min\{k \geq i+M : c_y = y_{[k,k+\mathrm{period}(y)[}\},$$

and note that i_0 is determined by $x_{[i-P,i+P[}$ since P is larger than a z-window. Similarly, set

$$j_0 = \max\{k \leq j-M : c_y = y_{[k,k+\mathrm{period}(y)[}\},$$

and note that j_0 is determined by $x_{[j-P,j+P[}$. Then $j_0 - i_0 \geq \#I - 2P - 2M$ and $j_0 - i_0 \in \mathrm{period}(y)\mathbb{N}$. The interval $[i_0, j_0[$ will be called a *special long marker interval* provided $j_0 - i_0 \geq T$, where

$$T = 2(L+R+D+M) + 2|u| + 6M.$$

When I is right-infinite, i.e. $I = [i,\infty[$ for some $i \in \mathbb{Z}$, we find that there is a unique $y \in \pi^{-1}(\mathcal{P})$ such that $w_j = y_j, j \geq i+P$, when $w = w_{i-P}w_{i-P+1}\dots w_{i+P-2}w_{i+P-1} \in \pi^{-1}\left(x_{[i-P,i+P[}\right)$. Set

$$i_0 = \min\{k \geq i+M : c_y = y_{[k,k+\mathrm{period}(y)[}\},$$

and note that i_0 is determined by $x_{[i-P,i+P[}$. The interval $[i_0,\infty)$ is a *right-infinite special marker interval.* Left-infinite special marker intervals are defined and constructed similarly. Note that different special marker intervals are at least a distance M apart, and that it only requires inspection of $x_{[i-T-2M,i+T+2M[}$ to decide if i is contained in a special long marker interval (finite or infinite).

Consider a special finite marker interval $[i_0, j_0[$, so that

$$j_0 - i_0 \geq 2(L+R+D+M) + 2|u| + 6M$$

by definition. Then $x_{[i_0,j_0[} = z_{[i_0,j_0[}$ for a unique $z \in \mathcal{P}$ by Lemma 3.2 of [B]. It follows from Lemma 7.7 that z is $\frac{\mathrm{period}(y)}{\mathrm{period}(z)}$-affiliated to X_0, and also that $\lambda(z)$ is $\frac{\mathrm{period}(y)}{\mathrm{period}(z)}$-affiliated to Y_0. There are therefore synchronizing words $v_z, u_z \in \mathbb{W}(Y_c)$ such that

$$u_z\left(\lambda(z)_{[j_0-\mathrm{period}(y),j_0[}\right)^k v_z = u_z\left(\lambda(z)_{[i_0,i_0+\mathrm{period}(y)[}\right)^k v_z \in \mathbb{W}(Y_c)$$

for all $k \in \mathbb{N}$. By Lemma 9.8 we can choose the lengths of u_z and v_z to be equal to R. Set

$$i_0' = \min\{i \geq i_0 + |u| + R + L : \; c_y = y_{[i,i+\mathrm{period}(y)[}\}.$$

We can then find a word $w \in \mathbb{W}_{i_0'-i_0-|u|-R}(Y_c)$ such that $uwu_z \in \mathbb{W}(Y_c)$. Note that $i_0' \leq i_0 + |u| + R + L + |c_y| \leq i_0 + |u| + R + L + P$. Similarly we can set

$$j_0' = \max\{j \leq j_0 - R - L : \; c_y = y_{[j-\mathrm{period}(y),j[}\},$$

and find a word $v \in \mathbb{W}_{j_0-j_0'-R}(Y_c)$ such that $v_z v u \in \mathbb{W}(Y_c)$. Note that $j_0' \geq j_0 - R - L - P$. We define $\varphi(x)_{[i_0,j_0[}$ to be the word

$$uwu_z\lambda(z)_{[i_0',j_0'[}v_z v.$$

This is a word in $\mathbb{W}(Y_c)$, because u_z and v_z are synchronizing for $\overline{Y_c}$ and $i_0' - i_0, j_0 - j_0' \in |c_y|\mathbb{N}$. Observe that we can choose u_z and w so that they only depend on $i_0' - i_0$ and $\lambda(z)_{[i_0,i_0+\mathrm{period}(y)[}$, and v_z and v so that they only depend on $j_0 - j_0'$ and $\lambda(z)_{[j_0-\mathrm{period}(y),j_0[}$. Set

$$T_0 = L + R + M + D,$$

and note that $\varphi(x)_{[i_0+|u|+T_0,j_0-T_0[}$ determines $\lambda(z)$ and hence z by Lemma 2.3 of [B], since $j_0 - T_0 - (i_0 + |u| + T_0) \geq 2P + 1$. We define $\varphi(x)_{[i_0,\infty[}$ and $\varphi(x)_{]-\infty,j_0[}$ in a similar way when $[i_0,\infty[$ and $]-\infty,j_0[$ are right-infinite and left-infinite special marker intervals, respectively. Finally, should $\mathbb{Z}$ be a special long marker interval, it follows that $x \in \mathcal{P}$, and we set $\varphi(x) = \lambda(x)$.

Note that it requires only inspection of $x_{[i-T-3M,i+T+3M[}$ to decide if $\varphi(x)_i$ has been defined (i.e. if i is contained in a special long marker interval), and to determine its value if it has.

To define $\varphi(x)_i$ for i in the remaining part of $\mathbb{Z}$ we apply Krieger's marker lemma as in [B]. By Lemma 2.2 of [B] (Krieger's marker lemma) there are a closed and open set $F \subseteq X$ and $k > 8M$ such that:

1) the sets $\sigma^i(F), 0 \leq i < M$, are disjont,
2) for any $i \in \mathbb{Z}$, if $x \in X$ and
$$\sigma^i(x) \notin \bigcup_{-M<j<M} \sigma^j(F),$$
then $x_{i-k}x_{i-k+1}\ldots x_{i+k}$ is a j-periodic word for some $j < M$, and
3) when $j < M$, and a j-periodic word of length $2k+1$ occurs in some x in X, then that word defines a j-periodic orbit which occurs in X.

Since F is closed and open, there is a number $|F| \in \mathbb{N}$ such that the condition $x \in F$ only depends on $x_{[-|F|,|F|[}$. For $x \in X$ the set of *marker coordinates* is the set $\mathcal{M}(x) = \{i \in \mathbb{Z} : \sigma^i(x) \in F\}$. A *finite marker interval* is a maximal interval $[i,j[$ in $\mathbb{Z}$ such that $k \notin \mathcal{M}(x)$ for all $i < k < j$. A *left-infinite marker interval* is a maximal interval of the form $]-\infty,j[$ containing no marker coordinates, and a *right-infinite marker interval* is a maximal interval of the form $[i,\infty[$ such that $]i,\infty[$ contains no marker coordinates.

The long marker intervals. We say that a marker interval I is *long* when $\#I \geq T$. When $[i,j[$ is a long finite marker interval, 2) and 3) imply that there is a unique periodic element $z \in X$ such that $x_{[i,j[} = z_{[i,j[}$; cf. Lemma 2.3 of [B]. Note that since $k > 8M$, it follows from 2) that every long marker interval is either contained in a special long marker interval, in which case $z \in \mathcal{P}$, or else the long marker interval has distance at least M to any special long marker interval, in which case $z \notin \mathcal{P}$. Consider a long marker interval such that the corresponding periodic point

z is not in $\mathcal{P}$. Then $z \in X_0^{(1)}$. Set

$$i_0 = \min\{i' \geq i + |u| + L + D : x_{[i', i'+\mathrm{period}(z)[} = p_z\},$$
$$j_0 = \max\{j' \leq j - L - D : x_{[j'-\mathrm{period}(z), j'[} = p_z\}.$$

Note that $j_0 - i_0 \in \mathrm{period}(z)\mathbb{N}$. We set

$$\varphi(x)_{[i,j[} = ue_- u_- (i_0 - i - |u| - L)\lambda(z)_{[i_0, j_0[} u_+ (j - j_0 - L)e_+.$$

Note that $\varphi(x)_{[i+T_0+|u|, j-T_0[}$ determines $\lambda(z)$ and hence z by Lemma 2.3 of [B], since $j - T_0 - (i + T_0 + |u|) \geq 2M + 1$. Left- or right-infinite long marker intervals are handled similarly, and if $\mathbb{Z}$ is a long (non-special) marker interval we set $\varphi(x) = \lambda(x)$.

Note that for any $i \in \mathbb{Z}$ it only requires inspection of $x_{[i-2T-|F|, i+2T+|F|[}$ to decide if $\varphi(x)_i$ has been defined, and to determine its value if it has.

The moderate marker intervals. Now consider a marker interval $[i, j[$ of length $j - i < T$ whose distance to a special marker interval is at least M. We set

$$\varphi(x)_{[i,j[} = ua_{x_{[i,j[}} c_{j-i}\left(x_{[i,j[}\right) b_{x_{[i,j[}}.$$

Since c_{j-i} is injective, $\varphi(x)_{[i+|u|+L, j-L[}$ determines $x_{[i,j[}$.

Note that for any $i \in \mathbb{Z}$ it still only requires inspection of $x_{[i-2T-|F|, i+2T+|F|[}$ to decide if $\varphi(x)_i$ has been defined, and to determine its value if it has.

The remaining intervals. We have now defined $\varphi(x)_i$ for all i outside a union of disjoint intervals—occurring at the ends of the special marker intervals—all of which have a length between M and $2T + 2M$. In such an interval $[i, j)$ we set

$$\varphi(x)_{[i,j[} = u^2 a'_{x_{[i,j[}} c'_{j-i}\left(x_{[i,j[}\right) b'_{x_{[i,j[}}.$$

Since c'_{j-i} is injective, $x_{[i,j[}$ is determined by $\varphi(x)_{[i+L+2|u|, j-L[}$.

Note that $\varphi(x) \in \overline{Y_c}$, because u is synchronizing for $\overline{Y_c}$. By construction φ commutes with the shift, and φ is continuous because $\varphi(x)_i$ is determined by $x_{[i-S, i+S[}$, where $S = 4T + 2M + |F|$.

Thanks to the careful choices, the position of the word u in $\varphi(x)$ gives away the intervals in x used to define $\varphi(x)$: An element of $\varphi(X)$ is made up by stretches of words from $\mathbb{W}\left(A_K \cup \lambda(\partial X)\right)$, all of length larger than $|u|$, and these stretches are only interrupted by stretches of the form aub or au^2b, where $\max\{|a|, |b|\} \leq L + D + R$. It follows that when $y = \varphi(x)$, then $i \in \mathbb{Z}$ is a left end-point of one of the intervals used in the definition if and only if $y_{[i, i+2|u|[} = u^2$ or $y_{[i, i+|u|[} = u$ and $y_{[i-|u|, i+2|u|[} \not\supseteq u^2$. Having identified the position of the intervals, we can recover x: If the interval $[i, j[$ begins with u^2 rather that u, it is one of the remaining intervals and we can recover $x_{[i,j[}$ since c'_{j-i} is injective. If the interval begins with u, but not u^2, and is of length $< T$, we can recover $x_{[i,j[}$ by using that c_{j-i} is injective. In the other intervals, beginning with u but not u^2, and all of length more than T, we simply use the middle piece $[i + T_0 + |u|, j - T_0[$ to identify the periodic point z of period $< M$ responsable for the interval. This uses that λ is injective, and the procedure does not require that we know if the interval arises from a special or non-special long marker interval. This shows that φ is injective. □

As pointed out above, the sufficient conditions of Theorem 9.9 for having an embedding $X \subseteq Y$ are almost also necessary. Specifically, when such an embedding exists, there must be an irreducible component Y_c of Y such that $h(X) \leq h\left(\overline{Y_c}\right)$ and $\mathrm{Per}\, X \hookrightarrow \mathrm{Per}\, \overline{Y_c}$. The discrepancy between the necessary and the sufficient

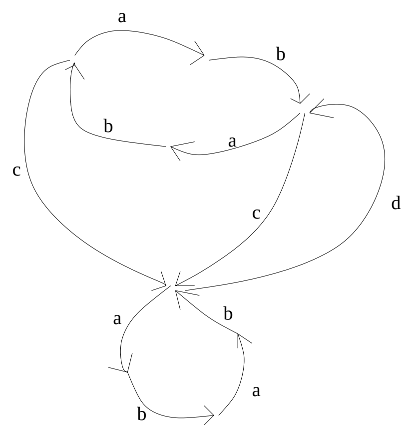


FIGURE 15.

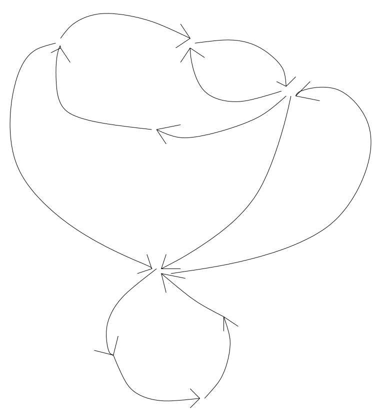
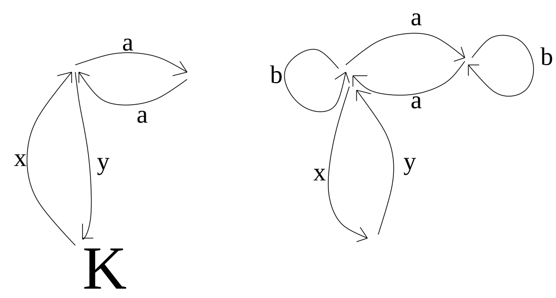


FIGURE 16.

conditions is of the same nature as with Theorem 8.6. Because of the structure of irreducible components in Y, this discrepancy does not disappear by requiring the embedding to be proper, as it does in the SFT case.

Example 9.10. We show by example that Theorem 9.9 is not true if X is only irreducible, rather than mixing. Let X be the strictly sofic shift space presented by the labeled graph of Figure 15, and Y the irreducible SFT given by the graph of Figure 16.

Note that X has transparent affiliation pattern since the only periodic orbit outside the top component, i.e. the orbit $(ab)^\infty$, has marked entries and exits. Furthermore, $h(X) < h(Y)$ and $\operatorname{Per} X \hookrightarrow \operatorname{Per} Y$. Nonetheless, X does not embed into Y; in fact, it follows from Proposition 8.1 that there are no morphisms from X into any non-mixing irreducible SFT. In particular, there are no morphisms $X \to Y$ at all. To see this, note that if X_c is the top component in X, then $\operatorname{period}(X_c) = 2$ by Lemma 6.21. Since $\operatorname{period}(Y) = 2$, all we need to know in order to conclude that Proposition 8.1 disallows any morphism $X \to Y$, is that the elements of the cyclic cover of X have non-empty intersection. To this end, observe that the words



The graph H

The graph G

FIGURE 17.

bc and cb are not equivalent as synchronizing words in X^2, and that they therefore represent the two different equivalence classes of synchronizing words in X^2. Since $cd(ab)^k cd$, $cdc(ab)^{2k} dcd$, $dc(ab)^{2k} dc$ and $dcd(ab)^k cdc$ are words in X^2 for all $k \in \mathbb{N}$, we conclude that the periodic orbit represented by $(ab)^\infty$ belongs to the closure of both of the two different irreducible components at level 0 in X^2, i.e. this periodic point is in the intersection of the two elements of the cyclic cover of X.

The obstruction in this example is of the same nature as in Example 3.5 of [B].

Example 9.11. In this example we show that Theorem 9.9 does not hold for general mixing sofic shift spaces X and Y; not even when Y has transparent affiliation pattern. Let K be a primitive graph which we consider as a labeled graph, labeled by natural numbers, such that different edges have different labels. Consider then the two labeled graphs H and G of Figure 17.

In H the edge labeled x terminates at the vertex in K where the edge labeled y starts. Let Y and X be the mixing sofic shift spaces presented by the labeled graphs H and G, respectively. We choose K such that

1) $q_1(X_K) = 1$,
2) $q_n(X_K) > q_n(X),\ n \geq 2$, and
3) $h(X_K) > h(X)$.

Then ∂Y consists of the fixed point a^∞ which is 2-affiliated, but *not* 1-affiliated, to the top component Y_c of Y. Since $\partial Y = \{a^\infty\}$, Y is an example of a mixing sofic shift space which is near Markov. In particular, Y has transparent affiliation pattern by Proposition 9.7. In X the derived shift space ∂X is the full 2-shift. All periodic points of ∂X are either 1- or 2-affiliated to the top component X_c of X. Since all periodic points of X_K are elements of the top component in Y, it follows from 2) that there is an injective map $\lambda_n : Q_n(X) \to Q_n(Y)$ such that $x \in X_c^{(d)} \Rightarrow \lambda_n(x) \in Y_c^{(1)} \subseteq Y_c^{(d)}$ for all $d \in \mathbb{N}$ when $n \geq 2$. Note that X has only the two fixed points a^∞ and b^∞. b^∞ is 1-affiliated to the top-component X_c of X, while a^∞ is d-affiliated to X_c exactly when d is even. The fixed point of Y arising from the condition that $q_1(X_K) = 1$ is 1-affiliated to Y_c, while the other fixed point, a^∞, in Y is d-affiliated to Y_c exactly when d is even. It follows that $\operatorname{Per} X \hookrightarrow \operatorname{Per} Y$. Nonetheless, X does not embed into Y. In fact, ∂X, which is the only irreducible component in X apart from the top component, does not embed into Y. To see this, observe that since ∂X is an irreducible SFT, it can only embed into Y if Y has a component with two fixed points 1-affiliated to it; cf. condition a) of Theorem 8.2. But the two fixed points of Y are not both 1-affiliated to the same component of Y.

Note that it is still the necessary conditions arising from Theorem 7.4 which prevent the embedding of X into Y in Example 9.11. The trouble seems to be that the condition spelled out in Definition 9.1 only gives conditions on the affiliation pattern of the periodic points to the top component, while Theorem 7.4 tells us that the affilation pattern to all irreducible components must be respected in order to have a morphism. Hence Example 9.11 does not exclude the possibility that all the necessary conditions arising from Theorem 7.4 can give us conditions that are also sufficient for the embedding of mixing sofic shift spaces of unequal entropies.

9.2. **A version Boyle's lower entropy factor theorem for sofic systems.** We now turn to the lower entropy factor theorem for mixing sofic shift spaces, when the domain shift has transparent affiliation pattern.

Definition 9.12. Let X and Y be irreducible sofic shift spaces with top components X_c and Y_c. We write $\operatorname{Per} X \searrow \operatorname{Per} Y$ when

$$\bigcap_{d\in F} Q_n\left(X_c^{(d)}\right) \neq \emptyset \Rightarrow \bigcup_{\{m\in\mathbb{N}:\ m|n\}} \left(\bigcap_{d\in F} Q_m\left(Y_c^{\left(\frac{nd}{m}\right)}\right)\right) \neq \emptyset$$

for every subset $F \subseteq \mathbb{N}$.

Observe that by Corollary 7.9 it suffices to check the condition of Definition 9.12 when F is a subset of $\{1, 2, \ldots, \operatorname{pm}(\varphi)\}$, where $\operatorname{pm}(\varphi)$ is the (periodic) multiplicity of the Fischer cover of X. It follows from Lemma 7.2 that Y can only be a factor of X when $\operatorname{Per} X \searrow \operatorname{Per} Y$. When X and Y are irreducible SFTs we have $\operatorname{Per} X = \operatorname{Per} X_c^{(1)}$ and $\operatorname{Per} Y = \operatorname{Per} Y_c^{(1)}$, so in this case the meaning of $\operatorname{Per} X \searrow \operatorname{Per} Y$ is the same as in [B] or [LM]. In particular, it follows from Boyle's factor theorem, Theorem 2.5 of [B], that the condition $\operatorname{Per} X \searrow \operatorname{Per} Y$ is sufficient for X to factor onto Y, when $h(Y) < h(X)$ and X and Y are both irreducible SFTs. By Theorem 3.3 of [B] the condition is also sufficient when $h(Y) < h(X)$, X is a sofic shift space and Y is a mixing inclusive sofic shift space. We extend Boyle's result to cover the case when X is an irreducible sofic shift space with transparent affiliation pattern, and Y is an arbitrary mixing sofic shift space.

Theorem 9.13. *Let X and Y be sofic shift spaces, Y mixing and X irreducible with transparent affiliation pattern. Assume that $h(X) > h(Y)$. Then Y is a factor of X if and only if* $\operatorname{Per} X \searrow \operatorname{Per} Y$.

Proof. As pointed out above, the condition that $\operatorname{Per} X \searrow \operatorname{Per} Y$ is necessary also when $h(X) = h(Y)$. We prove that it is sufficient when $h(X) > h(Y)$.

Let Z be a mixing SFT and $Z \to Y$ a finite-to-one factor map. Let X_c be the top component of X and $p = \operatorname{period}(X_c)$ the period of X_c. By Theorem 3.2 there is a sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ of irreducible SFTs inside X_c such that $\operatorname{period}(A_i) = p$ for all i, and $\lim_{n\to\infty} h(A_n) = h(X)$. Choose $D \in \mathbb{N}$ such that $h(A_D) > h(Y)$. Let $C_0, C_1, \ldots, C_{p-1}$ be the irreducible components of $(A_D)^p$, and consider C_0 as a shift space under the action of σ^p. Then $h(C_0) = ph(A_D) > h(Z^p)$. It follows that there is an $N \in \mathbb{N}$ so large that $q_n(C_0) > q_n(Z^p) + n$ for all $n \geq N$. By repeated applications of Boyle's 'blowing up' lemma; cf. Lemma 2.1 of [B] or Lemma 10.3.2 of [LM], we can then construct a mixing SFT W_0 such that

$h(W_0) = h(Z^p)$ and $q_n(C_0) \geq q_n(W_0)$ for all $n \in \mathbb{N}$, and a factor map $\varphi : W_0 \to Y^p$. By Krieger's embedding theorem, W_0 is a subshift of C_0, i.e. $W_0 \subseteq C_0 \subseteq X^p$. Set

$$W = W_0 \cup \sigma(W_0) \cup \sigma^2(W_0) \cup \cdots \cup \sigma^{p-1}(W_0).$$

Then W is an irreducible subshift of A_D, i.e. $W \subseteq A_D \subseteq X$, and there is a factor map $\varphi : W \to Y$. It remains only to construct an extension $\pi : X \to Y$ of φ. Thanks to Proposition 5.6 and the fact that the property of having transparent affiliation pattern is a conjugacy invariant, we can pass to a higher block presentation of X to arrange that φ becomes a one-block map, and that 1 becomes a step-length for W. When L_0 is a transition length for W_0, $L = pL_0$ has the property that for all words $x, y \in \mathbb{W}(W)$ there is a word $z \in \mathbb{W}_L(W)$ such that $xzy \in \mathbb{W}(W)$. By increasing L_0 if necessary we may assume that L is also a transition length for Y. Choose a word $u \in \mathbb{S}(Y)$ such that $u^2 \in \mathbb{W}(Y)$, and let $u_0 \in \varphi^{-1}(u) \cap \mathbb{W}(W)$. For each word $x \in \mathbb{W}(W)$ we can then choose words $a(\pm, x) \in \mathbb{W}_L(W)$ such that

$$u_0 a(-,x) x a(+,x) u_0 \in \mathbb{W}(W).$$

Let Y_c be the top component of Y. Since $\operatorname{Per} X \searrow \operatorname{Per} Y$, there is a shift-commuting map $\lambda : \operatorname{Per} X \to \operatorname{Per} Y$ such that

$$x \in Q_n\left(X_c^{(d)}\right) \Rightarrow \lambda(x) \in \bigcup_{\{m \in \mathbb{N}:\ m|n\}} Q_m\left(Y_c^{\left(\frac{nd}{m}\right)}\right). \tag{9.8}$$

Let $z \in X_c^{(1)}$ and choose a minimal cycle $p_z \in \mathbb{W}(X)$ such that $z = (p_z)^\infty$. We choose p_z such that $p_z = p_{z'}$ when z and z' are in the same orbit. Choose also $i \in \mathbb{Z}$ such that $z_{[i,i+\operatorname{period}(z)[} = p_z$. Let R be the number from Lemma 9.8 corresponding to Y. It follows from (9.8) that there are words $w_{(z,i)}, v_{(z,i)} \in \mathbb{S}(Y)$ of length R such that $u_{(z,i)}\left(\lambda(z)_{[i,i+\operatorname{period}(z)[}\right)^k v_{(z,i)} \in \mathbb{W}(Y)$ for all $k \in \mathbb{N}$. We can choose $u_{(z,i)}$ and $v_{(z,i)}$ such that they only depend on $\lambda(z)_{[i,i+\operatorname{period}(z)[}$. For each $k \geq L$ we choose also $u'_{(z,i,k)}, v'_{(z,i,k)} \in \mathbb{W}_k(Y)$ such that $uu'_{(z,i,k)}u_{(z,i)}, v_{(z,i)}v'_{(z,i,k)}u \in \mathbb{W}(Y)$. Note that $u'_{(z,i,k)}$ and $v'_{(z,i,k)}$ only depend on $\lambda(z)_{[i,i+\operatorname{period}(z)[}$ and k.

As in the proof Theorem 9.9, we let $\mathcal{P}$ be the periodic points of X that are not 1-affiliated to X_c. For each $y \in \pi^{-1}(\mathcal{P})$, choose a minimal cycle $c_y \in \mathbb{W}(X_G)$ so that $y = (c_y)^\infty$. Make the choice so that $c_y = c_{y'}$ when y and y' are in the same orbit. Let $i \in \mathbb{Z}$ be such that $y_{[i,i+\operatorname{period}(y)[} = c_y$. Then $z = \pi(y)$ is $\frac{\operatorname{period}(y)}{\operatorname{period}(z)}$-affiliated to X_c by Lemma 7.7. It follows from (9.8) that there are words $u_{(y,i)}, v_{(y,i)} \in \mathbb{S}(Y)$ such that

$$u_{(y,i)}\left(\lambda(\pi(y))_{[i,i+\operatorname{period}(y)[}\right)^k v_{(y,i)} \in \mathbb{W}(Y)$$

for all $k \in \mathbb{N}$. By Lemma 9.8 we can take $|u_z| = |v_z| = R$. Note that we can make the choices so that $u_{(y,i)}$ and $v_{(y,i)}$ only depend on $\lambda(\pi(y))_{[i,i+\operatorname{period}(y)[}$. For each $k \geq L$ we choose also $u'_{(y,i,k)}, v'_{(y,i,k)} \in \mathbb{W}_k(Y)$ such that $uu'_{(y,i,k)}u_{(y,i)}, v_{(y,i)}v'_{(y,i,k)}u \in \mathbb{W}(Y)$. Note that $u'_{(y,i,k)}$ and $v'_{(y,i,k)}$ only depend on $\lambda(\pi(y))_{[i,i+\operatorname{period}(z)[}$ and k.

Set $P_0 = \max\{\operatorname{period}(y) : y \in \pi^{-1}(\mathcal{P})\}$. Let P be both larger than P_0 and larger than a z-window for all $z \in \mathcal{P}$. Set $T = L + R + 2P + |u|$.

Let $\tilde{\varphi} : \mathbb{W}_m(X) \to \mathbb{W}_m(Y)$ be an arbitrary extension of $\varphi : \mathbb{W}_m(W) \to \mathbb{W}_m(Y)$. For every word in $w \in \mathbb{W}_m(X)$ we choose $b_{(+,w)}, b_{(-,w)} \in \mathbb{W}_L(Y)$ such that

$$ub_{(-,w)}\tilde{\varphi}(w)b_{(+,w)}u \in \mathbb{W}_{m+2L+2|u|}(Y).$$

Let $x \in X$. An interval $I \subseteq \mathbb{Z}$ will be called a *W-stretch in x* when $x_{[i,j[} \in \mathbb{W}(W)$ for all finite $\mathbb{Z}$-intervals $[i,j[\subseteq I$. Note that two different W-stretches in x, both maximal with respect to inclusion, must be disjoint because 1 is a step-length for W. Consider a maximal W-stretch I in x such that $\#I > 4T$. When $I = [i,j[$ is finite, set $I' = [i+T, j-T[$, and when $I = [i,\infty[$ or $I =]-\infty, j[$, set $I' = [i+T,\infty[$ and $I' =]-\infty, j-T[$, respectively. When $I = \mathbb{Z}$, set $I' = \mathbb{Z}$. The resulting intervals will be called *W-intervals*.

Consider a W-interval I in x, of length $\#I \geq 6T$. Assume first that I is finite, say $I = [i,j[$. Then

$$u_0 a_- x_{[i+|u|+L, i+2T[}, \ x_{[j-2T, j-|u|-L[} a_+ u_0 \in \mathbb{W}(W),$$

where a_+ and a_- are shorthand for $a(+, x_{[j-2T,j-|u|-L[})$ and $a(-, x_{[i+|u|+L,i+2T[})$, respectively, and we set

$$\pi(x)_{[i,j[} = \varphi(u_0 a_- x_{[i+|u|+L, j-|u|-L[} a_+ u_0).$$

If $]-\infty, j[$ is a left-infinite W-interval, we set

$$\pi(x)_{]-\infty,j[)} = \varphi\left(x_{]-\infty,j-|u|-L[} a_+ u_0\right).$$

If $[i,\infty[$ is a right-infinite W-interval, we set

$$\pi(x)_{[i,\infty[} = \varphi\left(u_0 a_- x_{[i+|u|+L,\infty[}\right).$$

If $]-\infty,\infty[$ is a maximal W-stretch in W, we set $\pi(x) = \varphi(x)$.

Note that for a given $i \in \mathbb{Z}$, it requires only inspection of $x_{[i-7T,i+7T[}$ to decide whether or not $\pi(x)_i$ has been defined by the preceding recipe, and to determine its value if it has.

A *maximal $\mathcal{P}$-stretch* in x is a (possibly infinite) interval $I \subseteq \mathbb{Z}$ such that $\#I \geq 2P_0 + 1$, $x_{[i,j[} \subseteq z$ for some $z \in \mathcal{P}$ when $i, j \in I$ and $i < j$, and such that no interval in x with these properties contains I properly. Since $W \subseteq X_c$ and no element of $\mathcal{P}$ is 1-affiliated to X_c, we conclude that any overlap between a maximal $\mathcal{P}$-stretch and a maximal W-stretch is of length no more than P_0. It follows from Lemma 2.3 of [B] that $x_I \subseteq z$ for a unique $z \in \mathcal{P}$ when I is a maximal $\mathcal{P}$-stretch in x, and that different maximal $\mathcal{P}$-stretches in x cannot overlap in more than $2P_0 + 1$ coordinates.

Every maximal $\mathcal{P}$-stretch I in x, of length $\#I > 2T$, gives rise to a *$\mathcal{P}$-interval* by removing an interval of length T at all ends, in the same way as maximal W-stretches gave rise to W-intervals. Consider a $\mathcal{P}$-interval I of length $\geq 6T$, and assume first that I is finite, say $I = [i,j[$. Since $T \geq P$ and P is larger than a z-window for all $z \in \mathcal{P}$, there is a unique $y \in \pi^{-1}(\mathcal{P})$ such that $w_{[i+P,j-P[} = y_{[i+P,j-P[}$ when $w = w_{i-2T} w_{i-2T+1} \dots w_{i+2T-2} w_{j+2T-1} \in \pi^{-1}\left(x_{[i-2T,j+2T[}\right)$. Set

$$i_0 = \min\{k \geq i + T : c_y = y_{[k,k+\mathrm{period}(y)[}\},$$

and note that i_0 is determined by $x_{[i-2T,i+2T[}$. Similarly, set

$$j_0 = \max\{k \leq j + T : c_y = y_{[k,k+\mathrm{period}(y)[}\},$$

and note that j_0 is determined by $x_{[j-2T,j+2T[}$. Then $j_0 - i_0 \geq \#I - 4T$ and $j_0 - i_0 \in \mathrm{period}(y)\mathbb{N}$. We set

$$\pi(x)_{[i,j[} = u u'_{(y,i_0,i_0-i-R-|u|)} u_{(y,i_0)} \lambda(z)_{[i_0,j_0[} v_{(y,j_0)} v'_{(y,j_0,j-j_0-R-|u|)} u.$$

This is a word in $\mathbb{W}(Y)$, because $u_{(y,i_0)}$ and $v_{(y,j_0)}$ are synchronizing for Y.

When I is a right-infinite $\mathcal{P}$-interval, i.e. $I = [i, \infty[$ for some $i \in \mathbb{Z}$, we find that there is a unique $y \in \pi^{-1}(\mathcal{P})$ such that $w_j = y_j, j \geq i + P$, when

$$w = w_{i-2T}w_{i-2T+1}\dots w_{i+2T-2}w_{i+2T-1} \in \pi^{-1}\left(x_{[i-2T,i+2T[}\right).$$

Set

$$i_0 = \min\{k \geq i + T : c_y = y_{[k,k+\text{period}(y)[}\},$$

and note that i_0 is determined by $x_{[i-2T,i+2T[}$. We set

$$\pi(x)_{[i,\infty[} = uu'_{(y,i_0,i_0-i-R-|u|)}u_{(y,i_0)}\lambda(z)_{[i_0,\infty[}.$$

Left-infinite $\mathcal{P}$-intervals are handled similarly, and if $\mathbb{Z}$ is a $\mathcal{P}$-interval we set $\pi(x) = \lambda(x)$.

Note that it still only requires inspection of $x_{[i-7T,i+7T[}$ to decide if $\pi(x)_i$ has been defined (i.e. if i is contained in a $\mathcal{P}$-interval or a W-interval), and to determine its value if it has. Note also that two intervals which are different, and both either $\mathcal{P}$- or W-intervals, must have distance at least T.

To define $\pi(x)_i$ for i in the remaining part of $\mathbb{Z}$ we apply Krieger's marker lemma as in [B]. There are a closed and open set $F \subseteq X$ and $k > 10T$ such that:

1) the sets $\sigma^i(F), 0 \leq i < T$, are disjont,
2) for any $i \in \mathbb{Z}$, if $x \in X$ and

$$\sigma^i(x) \notin \bigcup_{-T<j<T} \sigma^j(F),$$

then $x_{i-k}x_{i-k+1}\dots x_{i+k}$ is a j-periodic block for some $j < T$, and
3) when $j < T$, and a j-periodic word of length $2k+1$ occurs in some x in X, then that word defines a j-periodic orbit which occurs in X.

Since F is closed and open, there is a number $|F| \in \mathbb{N}$ such that the condition $x \in F$ only depends on $x_{[-|F|,|F|[}$. For $x \in X$ the *marker coordinates* are $\mathcal{M}(x) = \{i \in \mathbb{Z} : \sigma^i(x) \in F\}$. A *finite marker interval* is a maximal interval $[i, j[$ in $\mathbb{Z}$ such that $k \notin \mathcal{M}(x)$ for all $i < k < j$. A *left-infinite marker interval* is a maximal interval of the form $]-\infty, j[$ containing no marker coordinates, and a *right-infinite marker interval* is a maximal interval of the form $[i, \infty[$ such that $]i, \infty[$ contains no marker coordinates. We say that a finite marker interval $[i, j[$ is long when $j - i \geq 6T$ and $x_{[i,i+T[} \notin \mathbb{W}(W \cup \mathcal{P})$. Note that since $k > 10T$, it follows from 2) that all long marker intervals have a distance to any W-interval or $\mathcal{P}$-interval of length $\geq 6T$, which is at least T.

When $[i, j[$ is a long finite marker interval, there is a unique periodic element $z \in X_c^{(1)}$, not in W, such that $x_{[i,j[} = z_{[i,j[}$; cf. Lemma 2.3 of [B]. We set

$$i_0 = \min\{k \geq i + T : p_z = x_{[k,k+\text{period}(z)[}\}$$

and

$$j_0 = \max\{k \leq j - T : p_z = x_{[k-\text{period}(z),k[}\}.$$

Define

$$\pi(x)_{[i,j[} = uu'_{(z,i_0,i_0-i-R-|u|)}u_{(z,i_0)}\lambda(z)_{[i_0,j_0[}v'_{(z,j_0,j-j_0-R-|u|)}v_{(z,j_0)}u.$$

When $]-\infty, j[$ is left-infinite, we set

$$\pi(x)_{]-\infty,j[} = \lambda(z)_{]-\infty,j_0[}v_{(z,j_0)}v'_{(z,j_0,j-j_0-R-|u|)}u,$$

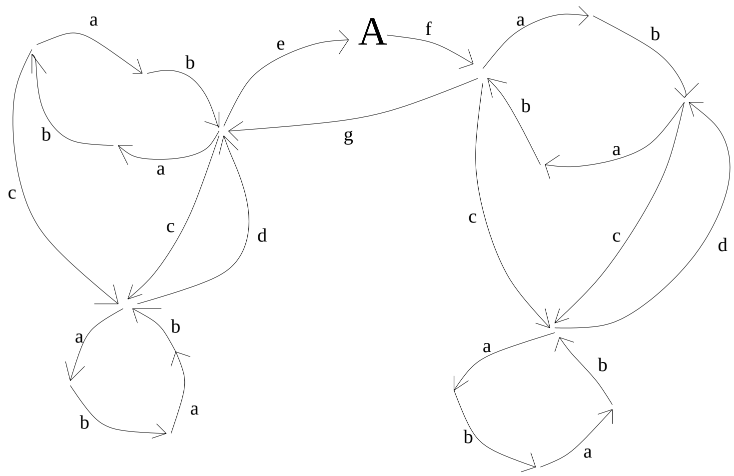


FIGURE 18.

where $j_0 = \max\{k \le j - T : x_{[k,k+\text{period}(z)[} = p_z\}$. Right-infinite long marker intervals are handled similarly, and should $\mathbb{Z}$ be a long marker interval we set $\pi(x) = \lambda(x)$.

For a given $i \in \mathbb{N}$ it still requires only knowledge of $x_{[i-7T-|F|,i+7T+|F|[}$ to decide if $\pi(x)_i$ has been defined, and determine its value if it has.

When $[i, j[$ is a marker interval with $j - i < 6T$ whose distance to any W-interval or $\mathcal{P}$-interval of length $\ge 6T$ is more than T, we set

$$\pi(x)_{[i,j[} = ub_{(-,w)}\tilde{\varphi}(x_{[i+L+|u|,j-L-|u|[})b_{(+,w)}u,$$

where $w = x_{[i+L+|u|,j-L-|u|[}$.

For a given $i \in \mathbb{N}$ it still requires only knowledge of $x_{[i-14T-|F|,i+14T+|F|[}$ to decide if $\pi(x)_i$ has been defined, and determine its value if it has.

We have now defined $\pi(x)_i$ for all i outside a union of non-overlapping intervals all of length between T and $7T$. In such an interval $[i, j[$, we set

$$\pi(x)_{[i,j[} = ub(+,w)\tilde{\varphi}(x_{[i+L+|u|,j-L-|u|[})b(+,w)u,$$

where $w = x_{[i+L+|u|,j-L-|u|[}$.

Since u is synchronizing for Y and $u^2 \in \mathbb{W}(Y)$, there is an element $\pi(x) \in Y$ with the prescribed coordinates, and by construction $\pi : X \to Y$ is a morphism, shift commuting by design and continuous because $\pi(x)_i$ only depends on $x_{[i-14T-|F|,i+14T+|F|[}$. π extends φ by construction and is therefore surjective since φ is. $\square$

Theorem 9.13 fails if Y is only irreducible rather than mixing. Indeed, the near Markov shift X of Example 9.10 does not factor onto Y when Y is only a period two orbit, although $\operatorname{Per} X \searrow \operatorname{Per} Y$ in this case. The obstruction of Proposition 8.1 strikes in here and prevents the existence of any morphism $X \to Y$.

Example 9.14. In this example we show that Theorem 9.13 is not true for general mixing sofic shift spaces X. To see this, consider the mixing sofic shift space X given by the labeled graph of Figure 18.

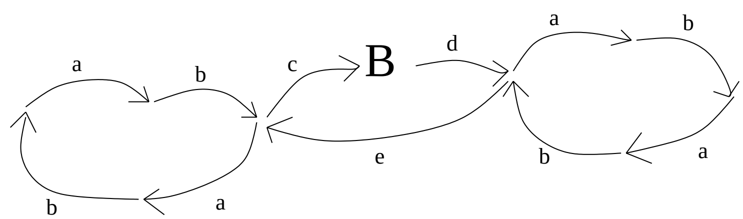


Figure 19.

Here A is an aperiodic graph, considered as a labeled graph with all edges labeled by different natural numbers, and the edges labeled e and f can terminate and initiate, respectively, at any vertex in A. A is chosen such that the edge shift X_A is a mixing SFT without fixed points or period two orbits, with an entropy $h(X_A)$ that exceeds the entropy of the sofic shift space Y we define below. Then ∂X is the irreducible non-mixing strictly sofic shift space considered in Example 9.10. As shown there, it follows from Proposition 8.1 that there are no morphisms from ∂X into any non-mixing SFT.

Let Y be the mixing sofic shift space given by the labeled graph of Figure 19.

Here B is an aperiodic graph, considered as a labeled graph with all edges labeled by different natural numbers. B is chosen such that X_B has no fixed points, no period two orbits, but orbits of any minimal period $n \geq 3$. Then $\operatorname{Per} X \searrow \operatorname{Per} Y$ and $h(X) > h(Y)$, but nonetheless, Y is not a factor of X. In fact, there is no morphism $\partial X \to Y$ whatsoever. To see this, note that all periodic points of ∂X are 1-affiliated to the top component of ∂X. If $\psi : \partial X \to Y$ is a morphism, Theorem 7.4 tells us that there is an irreducible component Y_c in Y such that $\psi(\partial X) \subseteq \overline{Y_c}$ and such that $\overline{Y_c}$ contains either a fixed point which is 2-affiliated to Y_c or else a period 2 orbit which is 1-affiliated to Y_c. Since Y contains no fixed points and only the period two orbit containing $(ab)^\infty$, which constitutes the only irreducible component in Y besides the top component, we can exclude the first possibility, i.e., Y_c must have a period 2 orbit 1-affiliated to it. Since $(ab)^\infty$ is not 1-affiliated to the top component of Y, we conclude that ψ must map ∂X into the period two orbit containing $(ab)^\infty$. As we saw above, this is impossible because a period 2 orbit is a non-mixing SFT.

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